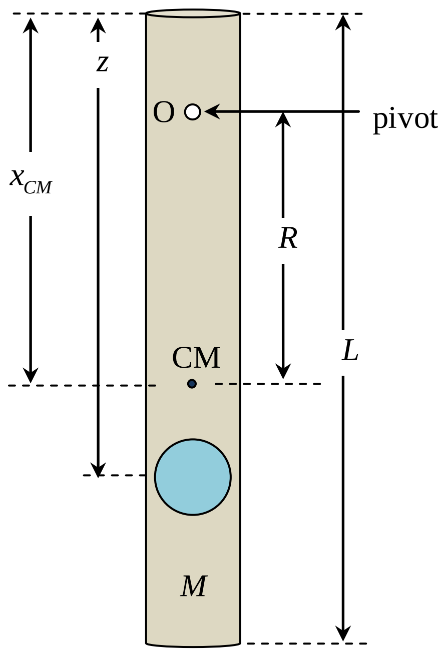
Solution: 2 . Mechanical Blackbox: a cylinder with a ball inside



In order to be able to calculate the required values in i, ii, iii, we need to know:

a. the position of the centre of mass of the tubing plus particle (object) which depends on z, m, M
b. the moment of inertia of the above.

The position of the CM may be found by balancing. The I_{CM} can be calculated from the period of oscillation of the tubing plus object.

Analytical steps to select parameters for plotting

I. $$x_{CM} = \frac{mz + M\frac{L}{2}}{m+M} \qquad \text{......................... (1)}$$

L is readily obtainable with a ruler.

x_{CM} is determined by balancing the tubing and object.

II. For small-amplitude oscillation about any point O the period T is given by considering the equation:

$$\left\{(M+m)R^2 + I_{CM}\right\}\ddot{\theta} = -g(M+m)R\sin\theta \approx -g(M+m)R\theta \qquad \text{................ (2)}$$

$$T = 2\pi\sqrt{\frac{I_{CM} + (M+m)R^2}{g(M+m)R}} \qquad \text{......................... (3)}$$

where
$$\begin{aligned} I_{CM} &= \frac{1}{3}M\left(\frac{L}{2}\right)^2 + M\left(x_{CM} - \frac{L}{2}\right)^2 + m\left(z - x_{CM}\right)^2 \\ &= \frac{1}{3}ML^2 + Mx_{CM}^2 - MLx_{CM} + m\left(z - x_{CM}\right)^2 \end{aligned} \qquad \text{......................... (4)}$$

Note that

$$T^2\frac{g(M+m)}{4\pi^2} = \frac{I_{CM}}{R} + (M+m)R \qquad \text{......................... (5)}$$

Method (a): (linear graph method)

The equation (5) may be put in the form:

$$T^2R = \left(\frac{4\pi^2}{g}\right)R^2 + \frac{4\pi^2 I_{CM}}{(M+m)g} \qquad \text{......................... (6)}$$

Hence the plot of T^2R v.s. R^2 will yield the straight line whose

$$\text{Slope} \quad \alpha = \frac{4\pi^2}{g} \qquad \text{......................... (7)}$$

$$\text{and y-intercept} \quad \beta = \frac{4\pi^2 I_{CM}}{(M+m)g} \qquad \text{......................... (8)}$$

$$\text{Hence,} \qquad I_{CM} = (M+m)\frac{\beta}{\alpha} \qquad \text{......................... (9)}$$

The value of g is from equation (7): $g = \dfrac{4\pi^2}{\alpha}$ .... (10)

Method (b): **minimum point curve method**

The equation (5) implies that T has a minimum value at

$$R = R_{\min} \equiv \sqrt{\frac{I_{CM}}{M+m}} \qquad \text{.........................} (11)$$

Hence $R_{\min}$ can be obtained from the graph T v.s. R.

And therefore

$$I_{CM} = (M+m)R_{\min}^2 \qquad \text{.........................} (12)$$

This equation (12) together with equation (1) will allow us to calculate the required values z and $\frac{M}{m}$.

At the value $R = R_{\min}$ equation (5) becomes $T_{\min}^2 \frac{g(M+m)}{4\pi^2} = (M+m)R_{\min} + (M+m)R_{\min}$

$$g = \frac{2R_{\min}}{T_{\min}^2} \times 4\pi^2 = \frac{8\pi^2 R_{\min}}{T_{\min}^2} \qquad \text{.........................} (13)$$

from which g can be calculated.

Results

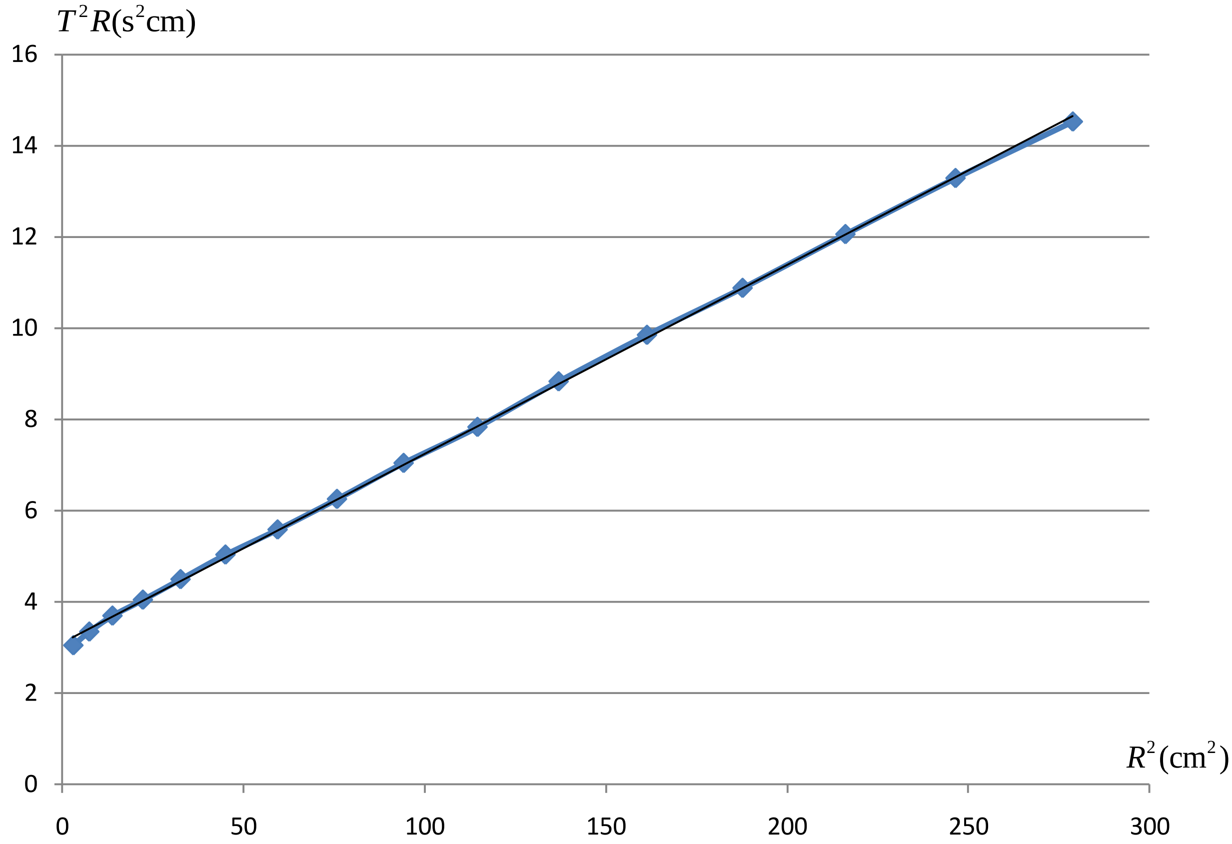
$$L = 30.0 \text{ cm} \pm 0.1 \text{ cm}$$

$$x_{CM} = 17.8 \text{ cm} \pm 0.1 \text{ cm (from top)}$$

$x_{CM} - R$ (cm)	time (s) for 20 cycles			T (s)	R (cm)	R^2 (cm^2)	T^2R (s^{2}cm)
1.1	18.59	18.78	18.59	0.933	16.7	278.9	14.53
2.1	18.44	18.25	18.53	0.920	15.7	246.5	13.29
3.1	18.10	18.09	18.15	0.906	14.7	216.1	12.06
4.1	17.88	17.78	17.81	0.891	13.7	187.7	10.88
5.1	17.69	17.50	17.65	0.881	12.7	161.3	9.85
6.1	17.47	17.38	17.28	0.869	11.7	136.9	8.83
7.1	17.06	17.06	17.22	0.856	10.7	114.5	7.83
8.1	17.06	17.00	17.06	0.852	9.7	94.1	7.04
9.1	16.97	16.91	16.96	0.847	8.7	75.7	6.25
10.1	17.00	17.03	17.06	0.852	7.7	59.3	5.58
11.1	17.22	17.37	17.38	0.866	6.7	44.9	5.03
12.1	17.78	17.72	17.75	0.888	5.7	32.5	4.49
13.1	18.57	18.59	18.47	0.927	4.7	22.1	4.04
14.1	19.78	19.90	19.75	0.991	3.7	13.7	3.69
15.1	11.16	11.13	11.13	1.114	2.7	7.3	3.34
16.1	13.25	13.40	13.50	1.338	1.7	2.9	3.04

Notes: at $x_{CM} - R = 15.1, 16.1$ cm, times for 10 cycles.

Method (a)



Calculation from straight line graph: slope $\alpha = 0.04108 \pm 0.0007$ s^2/cm, y-intercept $\beta = 3.10 \pm 0.05$ s^{2}cm

$$g = \frac{4\pi^2}{\alpha} \text{ giving } g = (961 \pm 20) \text{ cm/s}^2$$

$$\frac{\beta}{\alpha} = \frac{3.10}{0.04108} = 75.46 \text{ cm}^2 \left(\pm 2.5 \text{cm}^2\right)$$

$$I_{CM} = \left(M+m\right)\frac{\beta}{\alpha} = \left(75.46\right)\left(M+m\right)$$

From equation (4): $I_{CM} = \frac{1}{3}M\left(\frac{L}{2}\right)^2 + M\left(x_{CM} - \frac{L}{2}\right)^2 + m\left(z - x_{CM}\right)^2$

Then $(75.46)(M+m) = 75.0M + 7.84M + m(z-17.8)^2$

$$-7.38\frac{M}{m} + 75.46 = (z-17.8)^2 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (14)$$

The centre of mass position gives:

$$17.8(M+m) = 15.0M + mz$$

$$\frac{M}{m} = \frac{z-17.8}{2.8} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (15)$$

From equations (14) and (15):

$$-\frac{7.38}{2.8}(z-17.8) + 75.46 = (z-17.8)^2$$

$$(z-17.8) = 7.47$$

And $z = 25.27 = 25.3 \pm 0.1$ cm

$$\frac{M}{m} = 2.68 = 2.7$$

Error Estimation

Find error for g:

From (10), $g = \dfrac{4\pi^2}{\alpha}$

$$\Delta g = \frac{\Delta\alpha}{\alpha} g = 16.3\,\text{cm/s}^2 \approx 20\,\text{cm/s}^2$$

i) Find error for z:

First, find error for $r = \dfrac{\beta}{\alpha} = \dfrac{3.10}{0.04108} = 75.46\text{ cm}^2$.

$$\Delta r = \left(\frac{\Delta\alpha}{\alpha} + \frac{\Delta\beta}{\beta}\right) r = 2.5\,\text{cm}^2$$

Since error from r contributes most ($\dfrac{\Delta r}{r} \sim 0.03$ while $\dfrac{\Delta L}{L}, \dfrac{\Delta x_{cm}}{x_{cm}} \sim 0.005$), we estimate error propagation from r only to simplify the analysis by substituting the min and max values into equation (4).

Now, we use $r_{\max} = r + \Delta r = 75.46 + 2.5 = 77.96$. The corresponding quadratic equation is $(z-17.8)^2 + 1.743(z-17.8) - 77.96 = 0$ The corresponding solution is $(z-17.8)_{\max} = 7.55$ cm

If we use $r_{\min} = r - \Delta r = 75.46 - 2.5 = 72.96$, the corresponding quadratic equation is

$$(z-17.8)^2 + 3.529(z-17.8) - 72.96 = 0$$

The corresponding solution is $(z-17.8)_{\min} = 6.96$ cm

So $\Delta(z-17.8) = \dfrac{7.55-6.96}{2} = 0.3$ cm

Note that $\dfrac{\Delta(z-17.8)}{z-17.8} \sim 0.04$. So, we still ignore the error propagation due to $\Delta L, \Delta x_{cm}$

The error Δz can be estimated from $\Delta z \approx \Delta(z-17.8) = 0.3$ cm

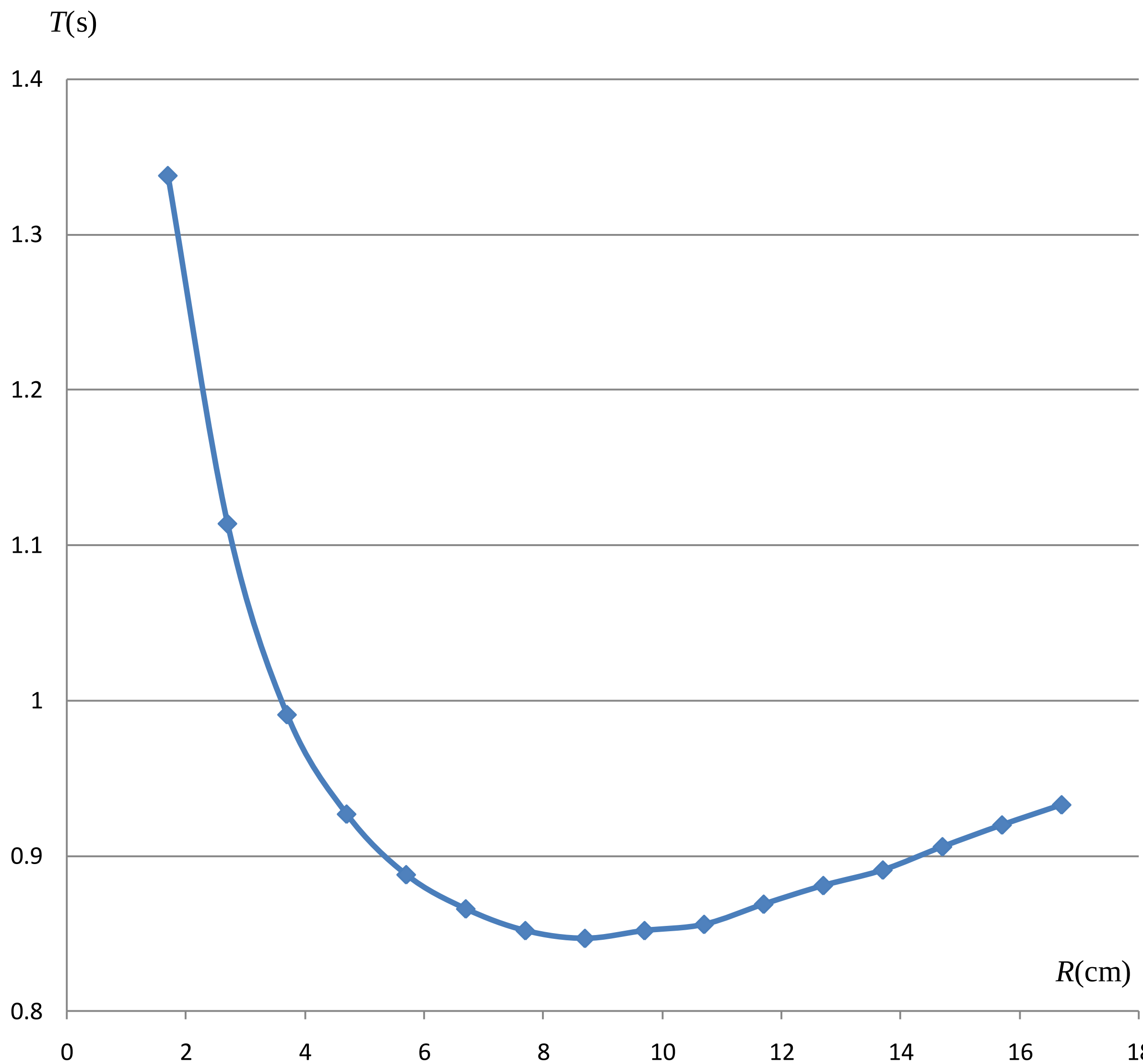
ii) Find error for $\dfrac{M}{m}$:

We know that $\dfrac{M}{m} = \dfrac{z-17.8}{2.8}$

$$\Delta\left(\frac{M}{m}\right) = \frac{\Delta(z-17.8)}{2.8} = 0.11$$

Method (b)

Calculation from *T*-*R* plot:



Using the minimum position: $T = T_{\min}$ at $I_{CM} = (M+m)R_{\min}^2$ and $g = \dfrac{8\pi^2 R_{\min}}{T_{\min}^2}$

From graph: $R_{\min} = 8.9 \pm 0.2$ cm and $T_{\min} = 0.846 \pm 0.005$ s

$$\therefore \quad g = 982 \pm 40 \text{ cm/s}^2$$

$$I_{CM} = (M+m)(8.9)^2 = (79.21)(M+m) \qquad \text{……………………. (16)}$$

From equations (14), (15), (16):

$$(79.21)(M+m) = 75.0M + 7.84M + m(z-17.8)^2$$

$$-3.63M + 79.21m = m(z-17.8)^2$$

$$(x-17.8)^2 + \frac{3.63}{2.8}(x-17.8) - 79.21 = 0$$

$$(z-17.8) = 8.28$$

And $z = 26.08 = 26.1 \pm 0.7$ cm

$$\frac{M}{m} = 2.95 = 3.0 \pm 0.3$$

<u>Error estimation</u>

i) Find error for g:

Using the minimum position: $g = \dfrac{8\pi^2 R_{\min}}{T_{\min}^2}$, we have

$$\Delta g = \left(\frac{\Delta R_{\min}}{R_{\min}} + 2\frac{\Delta T_{\min}}{T_{\min}}\right) g = 34 \approx 30\ \text{cm/s}^2$$

ii) Find error for z:

First, find error for $r = R_{\min}^2 = 79.21\ \text{cm}^2$.

$$\Delta r = 2R_{\min}\Delta R_{\min} = 3.56\ \text{cm}^2$$

This r is equivalent to r in part 1. So, one can follow the same error analysis.
As a result, we have

$$z = 26.08 \approx 26.1\ \text{cm}$$
$$\Delta z = 0.8\ \text{cm}$$

i) Find error for $\dfrac{M}{m}$:

Following the same analysis as in part I, we found that

$$\frac{M}{m} = 2.96\ ;\ \Delta\left(\frac{M}{m}\right) = 0.15$$

NOTE: This minimum curve method is not as accurate as the usual straight line graph.