

## Solution to Theoretical Question 3

### Part A

#### Neutrino Mass and Neutron Decay

(a) Let  $(c^2 E_e, c\vec{q}_e)$ ,  $(c^2 E_p, c\vec{q}_p)$ , and  $(c^2 E_\nu, c\vec{q}_\nu)$  be the energy-momentum 4-vectors of the electron, the proton, and the anti-neutrino, respectively, in the rest frame of the neutron. Notice that  $E_e, E_p, E_\nu, \vec{q}_e, \vec{q}_p, \vec{q}_\nu$  are all in units of mass. The proton and the anti-neutrino may be considered as forming a system of total rest mass  $M_c$ , total energy  $c^2 E_c$ , and total momentum  $c\vec{q}_c$ . Thus, we have

$$E_c = E_p + E_\nu, \quad \vec{q}_c = \vec{q}_p + \vec{q}_\nu, \quad M_c^2 = E_c^2 - q_c^2 \quad (\text{A1})$$

Note that the magnitude of the vector  $\vec{q}_c$  is denoted as  $q_c$ . The same convention also applies to all other vectors.

Since energy and momentum are conserved in the neutron decay, we have

$$E_c + E_e = m_n \quad (\text{A2})$$

$$\vec{q}_c = -\vec{q}_e \quad (\text{A3})$$

When squared, the last equation leads to the following equality

$$q_c^2 = q_e^2 = E_e^2 - m_e^2 \quad (\text{A4})$$

From Eq. (A4) and the third equality of Eq. (A1), we obtain

$$E_c^2 - M_c^2 = E_e^2 - m_e^2 \quad (\text{A5})$$

With its second and third terms moved to the other side of the equality, Eq. (A5) may be divided by Eq. (A2) to give

$$E_c - E_e = \frac{1}{m_n} (M_c^2 - m_e^2) \quad (\text{A6})$$

As a system of coupled linear equations, Eqs. (A2) and (A6) may be solved to give

$$E_c = \frac{1}{2m_n} (m_n^2 - m_e^2 + M_c^2) \quad (\text{A7})$$

$$E_e = \frac{1}{2m_n} (m_n^2 + m_e^2 - M_c^2) \quad (\text{A8})$$

Using Eq. (A8), the last equality in Eq. (A4) may be rewritten as

$$\begin{aligned} q_e^2 &= \frac{1}{2m_n} \sqrt{(m_n^2 + m_e^2 - M_c^2)^2 - (2m_n m_e)^2} \\ &= \frac{1}{2m_n} \sqrt{(m_n + m_e + M_c)(m_n + m_e - M_c)(m_n - m_e + M_c)(m_n - m_e - M_c)} \end{aligned} \quad (\text{A9})$$

Eq. (A8) shows that a maximum of  $E_e$  corresponds to a minimum of  $M_c^2$ . Now the rest mass  $M_c$  is the total energy of the proton and anti-neutrino pair in their center of mass (or momentum) frame so that it achieves the minimum

$$(M_c)_{\min} = M = m_p + m_v \quad (\text{A10})$$

when the proton and the anti-neutrino are both at rest in the center of mass frame. Hence, from Eqs. (A8) and (A10), the maximum energy of the electron  $E = c^2 E_e$  is

$$E_{\max} = \frac{c^2}{2m_n} [m_n^2 + m_e^2 - (m_p + m_v)^2] \approx 1.292569 \text{ MeV} \approx 1.29 \text{ MeV} \quad (\text{A11})$$

When Eq. (A10) holds, the proton and the anti-neutrino move with the same velocity  $v_m$  of the center of mass and we have

$$\frac{v_m}{c} = \left( \frac{q_v}{E_v} \right)_{E=E_{\max}} = \left( \frac{q_p}{E_p} \right)_{E=E_{\max}} = \left( \frac{q_c}{E_c} \right)_{E=E_{\max}} = \left( \frac{q_e}{E_c} \right)_{M_c=m_p+m_v} \quad (\text{A12})$$

where the last equality follows from Eq. (A3). By Eqs. (A7) and (A9), the last expression in Eq. (A12) may be used to obtain the speed of the anti-neutrino when  $E = E_{\max}$ . Thus, with  $M = m_p + m_v$ , we have

$$\begin{aligned} \frac{v_m}{c} &= \frac{\sqrt{(m_n + m_e + M)(m_n + m_e - M)(m_n - m_e + M)(m_n - m_e - M)}}{m_n^2 - m_e^2 + M^2} \\ &\approx 0.00126538 \approx 0.00127 \end{aligned} \quad (\text{A13})$$

### [Alternative Solution]

Assume that, in the rest frame of the neutron, the electron comes out with momentum  $c\vec{q}_e$  and energy  $c^2 E_e$ , the proton with  $c\vec{q}_p$  and  $c^2 E_p$ , and the anti-neutrino with  $c\vec{q}_v$  and  $c^2 E_v$ . With the magnitude of vector  $\vec{q}_\alpha$  denoted by the symbol  $q_\alpha$ , we have

$$E_p^2 = m_p^2 + q_p^2, \quad E_v^2 = m_v^2 + q_v^2, \quad E_e^2 = m_e^2 + q_e^2 \quad (\text{1A})$$

Conservation of energy and momentum in the neutron decay leads to

$$E_p + E_v = m_n - E_e \quad (\text{2A})$$

$$\vec{q}_p + \vec{q}_v = -\vec{q}_e \quad (\text{3A})$$

When squared, the last two equations lead to

$$E_p^2 + E_v^2 + 2E_p E_v = (m_n - E_e)^2 \quad (\text{4A})$$

$$q_p^2 + q_v^2 + 2\vec{q}_p \cdot \vec{q}_v = q_e^2 = E_e^2 - m_e^2 \quad (\text{5A})$$

Subtracting Eq. (5A) from Eq. (4A) and making use of Eq. (1A) then gives

$$m_p^2 + m_v^2 + 2(E_p E_v - \vec{q}_p \cdot \vec{q}_v) = m_n^2 + m_e^2 - 2m_n E_e \quad (\text{6A})$$

or, equivalently,

$$2m_n E_e = m_n^2 + m_e^2 - m_p^2 - m_v^2 - 2(E_p E_v - \vec{q}_p \cdot \vec{q}_v) \quad (7A)$$

If  $\theta$  is the angle between  $\vec{q}_p$  and  $\vec{q}_v$ , we have  $\vec{q}_p \cdot \vec{q}_v = q_p q_v \cos \theta \leq q_p q_v$  so that Eq. (7A) leads to the relation

$$2m_n E_e \leq m_n^2 + m_e^2 - m_p^2 - m_v^2 - 2(E_p E_v - q_p q_v) \quad (8A)$$

Note that the equality in Eq. (8A) holds only if  $\theta = 0$ , i.e., the energy of the electron  $c^2 E_e$  takes on its maximum value only when the anti-neutrino and the proton *move in the same direction*.

Let the speeds of the proton and the anti-neutrino in the rest frame of the neutron be  $c\beta_p$  and  $c\beta_v$ , respectively. We then have  $q_p = \beta_p E_p$  and  $q_v = \beta_v E_v$ . As shown in Fig. A1, we introduce the angle  $\phi_v$  ( $0 \leq \phi_v < \pi/2$ ) for the antineutrino by

$$q_v = m_v \tan \phi_v, \quad E_v = \sqrt{m_v^2 + q_v^2} = m_v \sec \phi_v, \quad \beta_v = q_v / E_v = \sin \phi_v \quad (9A)$$

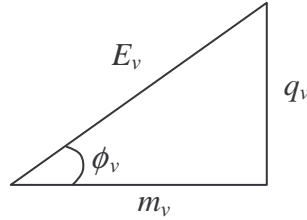


Figure A1

Similarly, for the proton, we write, with  $0 \leq \phi_p < \pi/2$ ,

$$q_p = m_p \tan \phi_p, \quad E_p = \sqrt{m_p^2 + q_p^2} = m_p \sec \phi_p, \quad \beta_p = q_p / E_p = \sin \phi_p \quad (10A)$$

Eq. (8A) may then be expressed as

$$2m_n E_e \leq m_n^2 + m_e^2 - m_p^2 - m_v^2 - 2m_p m_v \left( \frac{1 - \sin \phi_p \sin \phi_v}{\cos \phi_p \cos \phi_v} \right) \quad (11A)$$

The factor in parentheses at the end of the last equation may be expressed as

$$\frac{1 - \sin \phi_p \sin \phi_v}{\cos \phi_p \cos \phi_v} = \frac{1 - \sin \phi_p \sin \phi_v - \cos \phi_p \cos \phi_v}{\cos \phi_p \cos \phi_v} + 1 = \frac{1 - \cos(\phi_p - \phi_v)}{\cos \phi_p \cos \phi_v} + 1 \geq 1 \quad (12A)$$

and clearly assumes its minimum possible value of 1 when  $\phi_p = \phi_v$ , i.e., when the anti-neutrino and the proton *move with the same velocity* so that  $\beta_p = \beta_v$ . Thus, it follows from Eq. (11A) that the maximum value of  $E_e$  is

$$\begin{aligned} (E_e)_{\max} &= \frac{1}{2m_n} (m_n^2 + m_e^2 - m_p^2 - m_v^2 - 2m_p m_v) \\ &= \frac{1}{2m_n} [m_n^2 + m_e^2 - (m_p + m_v)^2] \end{aligned} \quad (13A)$$

and the maximum energy of the electron  $E = c^2 E_e$  is

$$E_{\max} = c^2 (E_e)_{\max} \approx 1.292569 \text{ MeV} \approx 1.29 \text{ MeV} \quad (14A)$$

When the anti-neutrino and the proton move with the same velocity, we have, from Eqs. (9A), (10A), (2A), (3A), and (1A), the result

$$\beta_v = \beta_p = \frac{q_p}{E_p} = \frac{q_v}{E_v} = \frac{q_p + q_v}{E_p + E_v} = \frac{q_e}{m_n - E_e} = \frac{\sqrt{E_e^2 - m_e^2}}{m_n - E_e} \quad (15A)$$

Substituting the result of Eq. (13A) into the last equation, the speed  $v_m$  of the anti-neutrino when the electron attains its maximum value  $E_{\max}$  is, with  $M = m_p + m_v$ , given by

$$\begin{aligned} \frac{v_m}{c} &= (\beta_v)_{\max E_e} = \frac{\sqrt{(E_e)_{\max}^2 - m_e^2}}{m_n - (E_e)_{\max}} = \frac{\sqrt{(m_n^2 + m_e^2 - M^2)^2 - 4m_n^2 m_e^2}}{2m_n^2 - (m_n^2 + m_e^2 - M^2)} \\ &= \frac{\sqrt{(m_n + m_e + M)(m_n + m_e - M)(m_n - m_e + M)(m_n - m_e - M)}}{m_n^2 - m_e^2 + M^2} \\ &\approx 0.00126538 \approx 0.00127 \end{aligned} \quad (16A)$$

## Part B

### Light Levitation

(b) Refer to Fig. B1. Refraction of light at the spherical surface obeys Snell's law and leads to

$$n \sin \theta_i = \sin \theta_t \quad (B1)$$

Neglecting terms of the order  $(\delta/R)^3$  or higher in sine functions, Eq. (B1) becomes

$$n\theta_i \approx \theta_t \quad (B2)$$

For the triangle  $\triangle FAC$  in Fig. B1, we have

$$\beta = \theta_t - \theta_i \approx n\theta_i - \theta_i = (n-1)\theta_i \quad (B3)$$

Let  $f_0$  be the frequency of the incident light. If  $n_p$  is the number of photons incident on the plane surface per unit area per unit time, then the total number of photons incident on the plane surface per unit time is  $n_p \pi \delta^2$ . The total power  $P$  of photons incident on the plane surface is  $(n_p \pi \delta^2)(hf_0)$ , with  $h$  being Planck's constant. Hence,

$$n_p = \frac{P}{\pi \delta^2 h f_0} \quad (B4)$$

The number of photons incident on an annular disk of inner radius  $r$  and outer radius  $r + dr$  on the plane surface per unit time is  $n_p (2\pi r dr)$ , where  $r = R \tan \theta_i \approx R\theta_i$ .

Therefore,

$$n_p (2\pi r dr) \approx n_p (2\pi R^2) \theta_i d\theta_i \quad (B5)$$

The  $z$ -component of the momentum carried away per unit time by these photons when

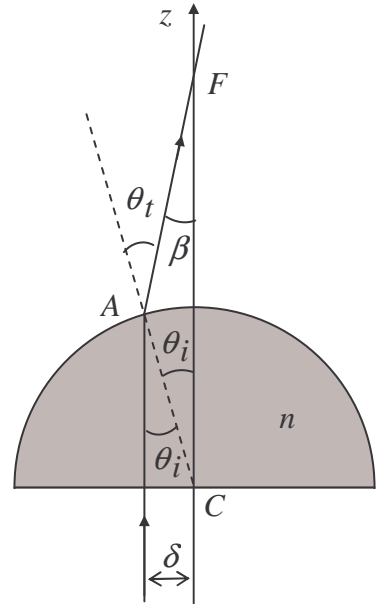


Fig. B1

refracted at the spherical surface is

$$\begin{aligned}
dF_z &= n_p \frac{hf_o}{c} (2\pi r dr) \cos \beta \approx n_p \frac{hf_o}{c} (2\pi R^2) \left(1 - \frac{\beta^2}{2}\right) \theta_i d\theta_i \\
&\approx n_p \frac{hf_o}{c} (2\pi R^2) \left[ \theta_i - \frac{(n-1)^2}{2} \theta_i^3 \right] d\theta_i
\end{aligned} \tag{B6}$$

so that the  $z$ -component of the total momentum carried away per unit time is

$$\begin{aligned}
F_z &= 2\pi R^2 n_p \left( \frac{hf_o}{c} \right) \int_0^{\theta_{im}} \left[ \theta_i - \frac{(n-1)^2}{2} \theta_i^3 \right] d\theta_i \\
&= \pi R^2 n_p \left( \frac{hf_o}{c} \right) \theta_{im}^2 \left[ 1 - \frac{(n-1)^2}{4} \theta_{im}^2 \right]
\end{aligned} \tag{B7}$$

where  $\tan \theta_{im} = \frac{\delta}{R} \approx \theta_{im}$ . Therefore, by the result of Eq. (B5), we have

$$F_z = \frac{\pi R^2 P}{\pi \delta^2 hf_o} \left( \frac{hf_o}{c} \right) \frac{\delta^2}{R^2} \left[ 1 - \frac{(n-1)^2 \delta^2}{4R^2} \right] = \frac{P}{c} \left[ 1 - \frac{(n-1)^2 \delta^2}{4R^2} \right] \tag{B8}$$

The force of optical levitation is equal to the sum of the  $z$ -components of the forces exerted by the incident and refracted lights on the glass hemisphere and is given by

$$\frac{P}{c} + (-F_z) = \frac{P}{c} - \frac{P}{c} \left[ 1 - \frac{(n-1)^2 \delta^2}{4R^2} \right] = \frac{(n-1)^2 \delta^2}{4R^2} \frac{P}{c} \tag{B9}$$

Equating this to the weight  $mg$  of the glass hemisphere, we obtain the minimum laser power required to levitate the hemisphere as

$$P = \frac{4mgcR^2}{(n-1)^2 \delta^2} \tag{B10}$$

## Marking Scheme

### Theoretical Question 3 Neutrino Mass and Neutron Decay

Total Scores	Sub Scores	Marking Scheme for Answers to the Problem
Part A 4.0 pts.	(a) 4.0	<p>The maximum energy of the electron and the corresponding speed of the anti-neutrino.</p> <ul style="list-style-type: none"> <li>➤ 0.5 use energy-momentum conservation and can convert it into equations.</li> <li>➤ 0.5 obtain an expression for <math>E_e</math> that allows the determination of its maximum value.</li> <li>➤ (0.5+0.2) for concluding that proton and anti-neutrino must move with the same velocity when <math>E_e</math> is maximum. (0.2 for the same direction)</li> <li>➤ 0.6 for establishing the minimum value of <math>(E_p E_v - \vec{q}_p \cdot \vec{q}_v)</math> to be <math>m_p m_v</math> or a conclusion equivalent to it.</li> <li>➤ (0.5+0.1) for expression and value of <math>E_{\max}</math>.</li> <li>➤ 0.5 for concluding <math>\beta_v = \sqrt{E_e^2 - m_e^2} / (m_n - E_e)</math>.</li> <li>➤ (0.5+0.1) for expression and value of <math>v_m/c</math>.</li> </ul>

### Light Levitation

Part B 4.0 pts	(b) 4.0	<p>Laser power needed to balance the weight of the glass hemisphere.</p> <ul style="list-style-type: none"> <li>➤ 0.3 for law of refraction <math>n \sin \theta_i = \sin \theta_t</math>.</li> <li>➤ 0.3 for making the linear approximation <math>n \theta_i \approx \theta_t</math>.</li> <li>➤ 0.4 for relation between angles of deviation and incidence.</li> <li>➤ 0.3 for photon energy <math>\varepsilon = h\nu</math>.</li> <li>➤ 0.3 for photon momentum <math>p = \varepsilon/c</math>.</li> <li>➤ 0.3 for momentum of incident photons per unit time <math>= P/c</math>.</li> <li>➤ 0.6 for momentum of photons refracted per unit time as a function of the angle of incidence.</li> <li>➤ 0.4 for total momentum of photons refracted per unit time <math>= [1 - (n-1)^2 \delta^2 / (4R^2)] P/c</math>.</li> <li>➤ 0.4 for force of levitation = sum of forces exerted by incident and refracted photons.</li> <li>➤ 0.4 for force of levitation <math>= (n-1)^2 \delta^2 P / (4cR^2)</math>.</li> <li>➤ 0.3 for the needed laser power <math>P = 4mgcR^2 / (n-1)^2 \delta^2</math>.</li> </ul>
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