THEORETICAL COMPETITION
Tuesday, July 23rd, 2002

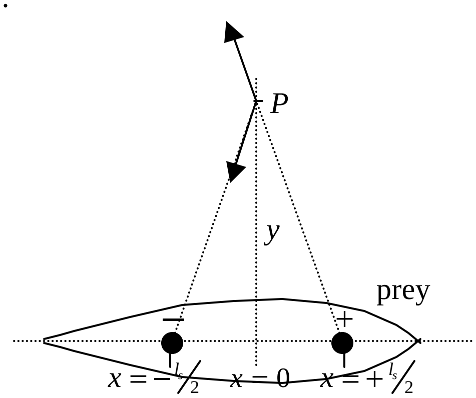
Solution II: Sensing Electrical Signals

1. When a point current source I_s is in infinite isotropic medium, the current density vector at a distance r from the point is

$$\vec{j} = \frac{I_s}{4pr^3}\vec{r}$$

[+1.5 pts] (without vector notation, -0.5 pts)

2.



Assuming that the resistivities of the prey body and that of the surrounding seawater are the same, implying the elimination of the boundary surrounding the prey, the two spheres seem to be in infinite isotropic medium with the resistivity of r. When a small sphere produces current at a rate I_s, the current flux density at a distance r from the sphere's center is also

$$\vec{j} = \frac{I_s}{4pr^3}\vec{r}$$

The seawater resistivity is r, therefore the field strength at r is

$$\vec{E}(\vec{r}) = r\vec{j} = \frac{rI_s}{4pr^3}\vec{r} \qquad \text{[+0.2 pts]}$$

In the model, we have two small spheres. One is at positive voltage relative to the other therefore current I_s flows from the positively charged sphere to the negatively charged sphere. They are separated by l_s. The field strength at P(0,y) is:

$$\vec{E}_p = \vec{E}_+ + \vec{E}_- \qquad [+0.8 \text{ pts}]$$

$$= \frac{rI_s}{4p}\left[\frac{1}{\left(\left(\frac{l_s}{2}\right)^2 + y^2\right)^{\frac{3}{2}}}\left(-\frac{l_s}{2}i + yj\right) + \frac{1}{\left(\left(\frac{l_s}{2}\right)^2 + y^2\right)^{\frac{3}{2}}}\left(-\frac{l_s}{2}i - yj\right)\right]$$

$$= \frac{rI_s}{4p}\left[\frac{l_s(-i)}{\left(\left(\frac{l_s}{2}\right)^2 + y^2\right)^{\frac{3}{2}}}\right]$$

$$\vec{E}_p \approx \frac{rI_s l_s}{4py^3}(-i) \quad \text{for ls << y} \qquad [+1.0 \text{ pts}]$$

3. The field strength along the axis between the two source spheres is:

$$\vec{E}(x) = \frac{rI_s}{4p}\left(\frac{1}{\left(x - \frac{l_s}{2}\right)^2} + \frac{1}{\left(x + \frac{l_s}{2}\right)^2}\right)(-i) \qquad [+0.5 \text{ pts}]$$

The voltage difference to produce the given current I_s is

$$V_s = \Delta V = V_+ - V_- = -\int_{\left(-\frac{l_s}{2}+r_s\right)}^{\left(\frac{l_s}{2}-r_s\right)} \vec{E}(x).d\vec{x} = -\frac{rI_s}{4p}\int\left(\frac{1}{\left(x - \frac{l_s}{2}\right)^2} + \frac{1}{\left(x + \frac{l_s}{2}\right)^2}\right)(-i).(idx) \quad [+0.5 \text{ pts}]$$

$$= \frac{rI_s}{4p}\left[\frac{1}{-2+1}\left(\frac{1}{\left(\frac{l_s}{2} - r_s - \frac{l_s}{2}\right)} - \frac{1}{\left(-\frac{l_s}{2} + r_s - \frac{l_s}{2}\right)}\right) + \frac{1}{-2+1}\left(\frac{1}{\left(\frac{l_s}{2} - r_s + \frac{l_s}{2}\right)} - \frac{1}{\left(-\frac{l_s}{2} + r_s + \frac{l_s}{2}\right)}\right)\right]$$

$$= \frac{rI_s}{4p}\left(\frac{2}{r_s} - \frac{2}{l_s - r_s}\right) = \frac{2rI_s}{4p}\left(\frac{l_s - r_s - r_s}{(l_s - r_s)r_s}\right) = \frac{rI_s}{2pr_s}\left(\frac{l_s - 2r_s}{l_s - r_s}\right)$$

$$V_s = \Delta V \approx \frac{rI_s}{2pr_s} \quad \text{for } l_s >> r_s. \qquad [+0.5 \text{ pts}]$$

The resistance between the two source spheres is:

$$R_s = \frac{V_s}{I_s} = \frac{\rho}{2\pi r_s}$$

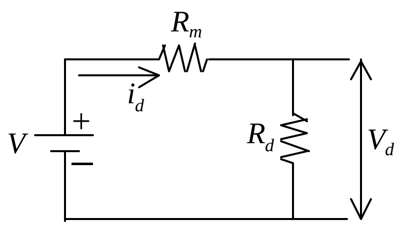
[+0.5 pts]

The power produced by the source is:

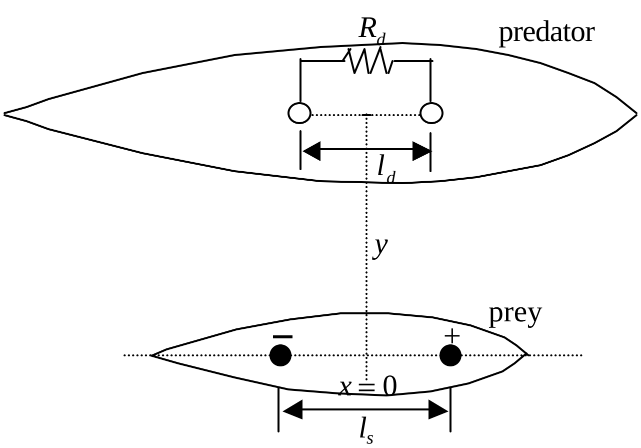
$$P = I_s V_s = \frac{\rho I_s^{\,2}}{2\pi r_s}$$

[+0.5 pts]

4.



V is the voltage difference between the detector's spheres due to the electric field induced by the prey, R_m is the inner resistance due to the surrounding sea water. V_d and R_d are respectively the voltage difference between the detecting spheres and the resistance of the detecting element within the predator and i_d is the current flowing in the closed circuit.



Analog to the resistance between the two source spheres, the resistance of the medium with resistivity ρ between the detector spheres, each having a radius of r_d is:

$$R_m = \frac{\rho}{2\pi r_d}$$

[+0.5 pts]

Since l_d is much smaller than y, the electric field strength between the detector spheres can be assumed to be constant, that is:

$$E = \frac{\rho I_s l_s}{4\pi y^3}$$ [+0.2 pts]

Therefore, the voltage difference present in the medium between the detector spheres is:

$$V = E l_d = \frac{\rho I_s l_s l_d}{4\pi y^3}$$ [+0.3 pts]

The voltage difference across the detector spheres is:

$$V_d = V\frac{R_d}{R_d + R_m} = \frac{rI_s l_s l_d}{4py^3}\frac{R_d}{R_d + \frac{r}{2pr_d}}$$

[+0.5 pts]

The power transferred from the source to the detector is:

$$P_d = i_d V_d = \frac{V}{R_d + R_m}V_d = \left(\frac{rI_s l_s l_d}{4py^3}\right)^2 \frac{R_d}{\left(R_d + \frac{r}{2pr_d}\right)^2}$$

[+0.5 pts]

5. P_d is maximum when

$$R_t = \frac{R_d}{\left(R_d + \frac{r}{2pr_d}\right)^2} = \frac{R_d}{\left(R_d + R_m\right)^2} \quad \text{is maximum} \qquad \text{[+0.5 pts]}$$

Therefore,

$$\frac{dR_t}{dR_d} = \frac{1\left(R_d + R_m\right)^2 - R_d 2\left(R_d + R_m\right)}{\left(R_d + R_m\right)^4} = 0 \qquad \text{[+0.5 pts]}$$

$$\left(R_d + R_m\right) - 2R_d = 0$$

$$R_d^{optimum} = R_m = \frac{r}{2pr_d} \qquad \text{[+0.5 pts]}$$

The maximum power is:

$$P_d^{\max imum} = \left(\frac{rI_s l_s l_d}{4py^3}\right)^2 \frac{pr_d}{2\,r} = \frac{r\left(I_s l_s l_d\right)^2 r_d}{32py^6}$$

[+0.5 pts]