# $29^{th}$ International Physics Olympiad
# Reykjavík, Iceland

## Theoretical competition

Saturday, July $4^{th}$, 1998

**9 a.m. – 2 p.m.**

**Read this first:**

1. Use only the pen provided.
2. Use only the front side of the answer sheets.
3. Use *as little text as possible* in your answers; express yourself primarily with equations, numbers and figures. **Summarize your results on the answer sheets.**
4. For anything but your answers and your graphs use the blank answer sheets. This applies e.g. when you are asked to *show that* ... and also for all calculations you want to be considered for evaluation.
5. You may often be able to solve later parts of a problem without having solved the previous ones. In such cases you may take the result of a previous part as given, in the form stated in the problem text.
6. Please indicate on all sheets your team name, student number, number of page and total number of pages. On the blank answer sheets also indicate the problem number.
7. At the end of the exam please put your answer sheets in order. You may leave on your table material which you do not wish to be evaluated.

**This set of problems consists of 11 pages in total.**

Examination prepared at:
University of Iceland, Department of Physics, in collaboration with physicists from the National Energy Authority.

# 1 Rolling of a hexagonal prism[1]

## 1.1 Problem text

Consider a long, solid, rigid, regular hexagonal prism like a common type of pencil (Figure 1.1). The mass of the prism is $M$ and it is uniformly distributed. The length of each side of the cross-sectional hexagon is $a$. The moment of inertia $I$ of the hexagonal prism about its central axis is

$$I = \frac{5}{12}Ma^2 \tag{1.1}$$

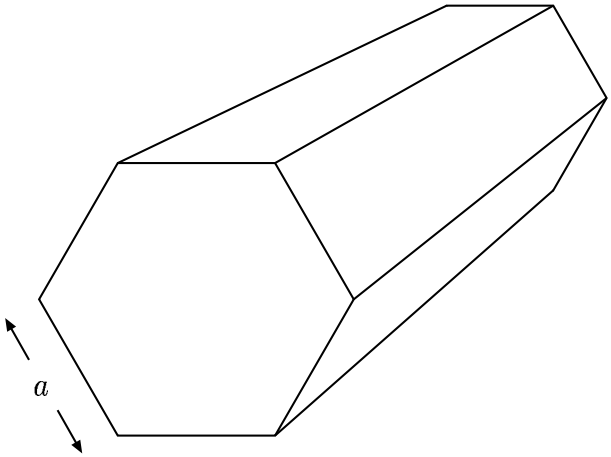


Figure 1.1: *A solid prism with the cross section of a regular hexagon.*

The moment of inertia $I'$ about an edge of the prism is

$$I' = \frac{17}{12}Ma^2 \tag{1.2}$$

a) (3.5 points) The prism is initially at rest with its axis horizontal on an inclined plane which makes a small angle $\theta$ with the horizontal (Figure 1.2). Assume that the surfaces of the prism are slightly concave so that the prism only touches the plane at its edges. The effect of this concavity on the moment of inertia can be ignored. The prism is now displaced from rest and starts an uneven rolling down the plane. Assume that friction prevents any sliding and that the prism does not lose contact with the plane. The angular velocity just before a given edge hits the plane is $\omega_i$ while $\omega_f$ is the angular velocity immediately after the impact.

Show that we may write

$$\omega_f = s\omega_i \tag{1.3}$$

and write the value of the coefficient $s$ on the answer sheet.

[1]Authors: Leó Kristjánsson and Thorsteinn Vilhjálmsson

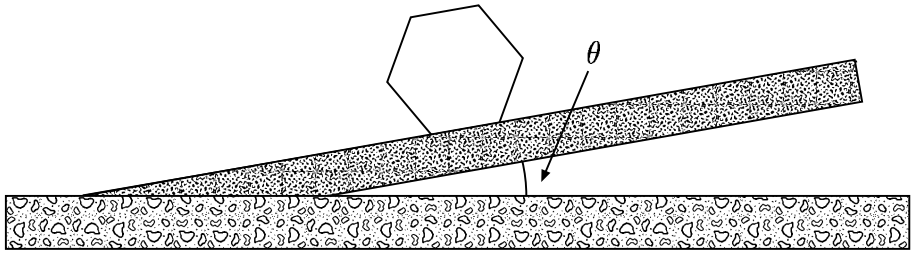


Figure 1.2: *A hexagonal prism lying on an inclined plane.*

b) (1 point) The kinetic energy of the prism just before and after impact is similarly $K_i$ and $K_f$.

Show that we may write

$$K_f = rK_i \tag{1.4}$$

and write the value of the coefficient $r$ on the answer sheet.

c) (1.5 points) For the next impact to occur $K_i$ must exceed a minimum value $K_{i,min}$ which may be written in the form

$$K_{i,min} = \delta Mga \tag{1.5}$$

where $g = 9.81$ m/s$^2$ is the acceleration of gravity.

Find the coefficient $\delta$ in terms of the slope angle $\theta$ and the coefficient $r$. Write your answer on the answer sheet. (Use the algebraic symbol $r$, not its value).

d) (2 points) If the condition of part (c) is satisfied, the kinetic energy $K_i$ will approach a fixed value $K_{i,0}$ as the prism rolls down the incline.

Given that the limit exists, show that $K_{i,0}$ may be written as:

$$K_{i,0} = \kappa Mga \tag{1.6}$$

and write the coefficient $\kappa$ in terms of $\theta$ and $r$ on the answer sheet.

e) (2 points) Calculate, to within 0.1°, the minimum slope angle $\theta_0$, for which the uneven rolling, once started, will continue indefinitely. Write your numerical answer on the answer sheet.

## 1.2 Solution

a)
*Solution Method 1*

At the impact the prism starts rotating about a new axis, i.e. the edge which just hit the plane. The force from the plane has no torque about this axis, so that the angular momentum about the edge is conserved during the brief interval of impact. The linear

momentum of the prism as a whole has the same direction as the velocity of the center of mass ($\vec{P} = M\ \vec{v}_C$ where the subscript $C$ refers to the center of mass), and this direction is easy to follow when we know the axis of rotation at a given time. Just before impact $\vec{P}$ is directed 30° downwards relative to the plane, but will after impact point 30° upwards from the plane, see Figure 1.3.

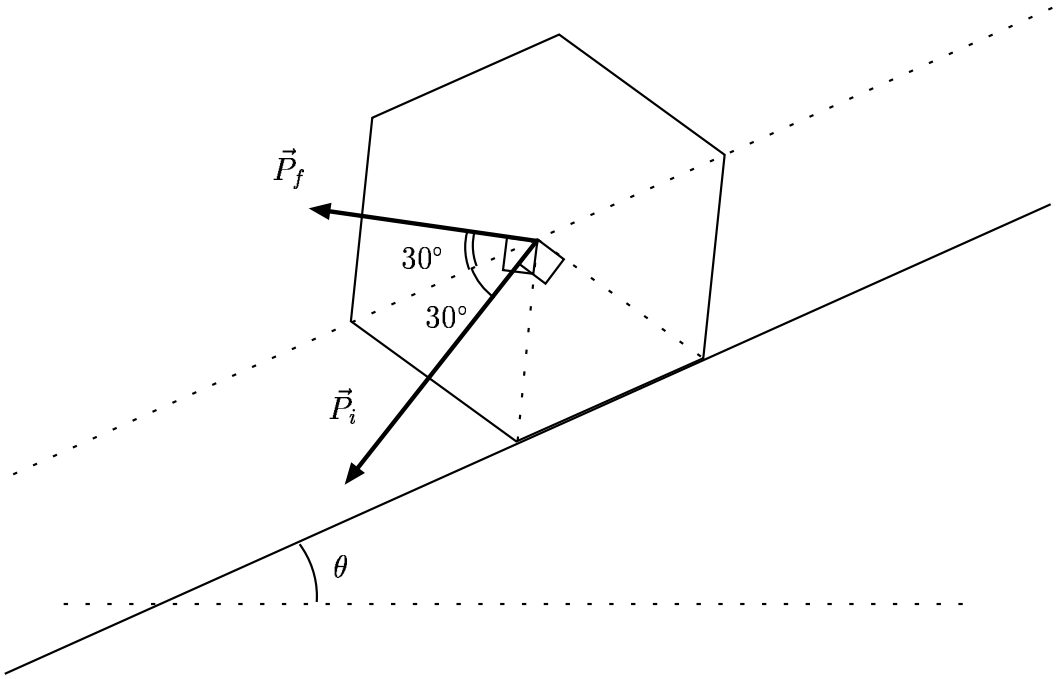


Figure 1.3: *The linear momentum of the prism as a whole, before and after impact.*

To find the angular momentum about the edge of impact just before the impact we use the equation relating angular momentum $\vec{L}$ about an arbitrary axis to the angular momentum $\vec{L}_C$ about an axis through the center of mass parallel to the first one:

$$\vec{L} = \vec{L}_C + M\ \vec{r}_C \times \vec{v}_C \tag{1.7}$$

where the subscript $C$ refers to the center of mass. Here, this is applied to an axis at the point of impact so that $\vec{r}_C$ is the vector from that point to the center of mass (Figure 1.3). The vectors on the right hand side of equation (1.7) both have the same direction. Hence we get for the quantities just before impact[2]

$$|\vec{r}_C \times \vec{v}_{Ci}| = r_C\ v_{Ci} \sin 30° = a^2\ \omega_i\ /\ 2 \tag{1.8}$$

$$L_i = I\ \omega_i + \frac{1}{2}\ M\ a^2\ \omega_i = \left(\frac{5}{12} + \frac{1}{2}\right)\ M\ a^2\ \omega_i = \frac{11}{12}\ M\ a^2\ \omega_i \tag{1.9}$$

On the other hand, angular momentum about the edge just after impact is, from equation (1.2):[3]

[2]This may also be done by using Steiner's theorem twice, going from the previous axis of impact to the center of mass and from there to the new axis of impact.

[3]Alternatively:

$$L_f = I'\omega_f = \frac{17}{12} M a^2 \omega_f \tag{1.10}$$

where the subscript $f$ always refers to the situation just after impact. We may notice that the difference comes about because of the different directions of $\vec{v}_{Ci}$ and $\vec{v}_{Cf}$. Now, when we state the conservation of angular momentum, $L_i = L_f$, we obtain a relation between the angular velocities as follows:

$$\omega_f = \frac{11/12}{17/12}\, \omega_i = \frac{11}{17}\, \omega_i \tag{1.11}$$

We thus get:

$$\boldsymbol{s = 11/17} \tag{1.12}$$

We may note that $s$ is independent of $a$, $\omega_i$, and $\theta$.

*Solution Method 2*

On impact the prism receives an impulse $\vec{P}$ [N · s] from the plane at the edge where the impact occurs. There is no reaction at the edge which is leaving the plane. The impulse has a component $P_{||}$ parallel to the inclined plane (positive upwards along the incline in Figure 1.3 and a component $P_\perp$ perpendicular to the plane (positive upwards from the plane in the same figure).

We can set up three equations with the three unknowns $P_{||}$, $P_\perp$ and the ratio $s = \frac{\omega_f}{\omega_i}$. The quantity $P_{||}$ is the change in the parallel component of the linear momentum of the prism and $P_\perp$ is the corresponding change in perpendicular linear momentum. Thus:

$$P_{||} = M\,(\omega_i - \omega_f)\, a \cdot \frac{\sqrt{3}}{2} \tag{1.13}$$

$$P_\perp = M\,(\omega_i + \omega_f)\, a \cdot \frac{1}{2}. \tag{1.14}$$

We finally have:

$$P_\perp a \frac{1}{2} - P_{||} a \frac{\sqrt{3}}{2} = I\,(\omega_i - \omega_f) \tag{1.15}$$

since the right hand side is the change in angular momentum about the center of mass. Equations (1.13), (1.14) and (1.15) can now be solved for the ratio $s = \frac{\omega_f}{\omega_i}$ giving, of course, the same result as before.

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$$\begin{aligned} L_f &= I\,\omega_f + M\,|\vec{r}_C \times \vec{v}_{Cf}| = I\omega_f + Ma^2\,\omega_f \sin 90^\circ \\ &= \left(\frac{5}{12} + 1\right)\, Ma^2\,\omega_f = \frac{17}{12}\, Ma^2\,\omega_f \end{aligned}$$

b)

The linear speed of the center of mass just before impact is $a\omega_i$ and just after impact it is $a\omega_f$. We know that we can always write the kinetic energy of a rotating rigid body as a sum of „internal“ and „external“ kinetic energy:

$$K_{tot} = \frac{1}{2}\ I\ \omega^2 + \frac{1}{2}\ M\ v_C^2 \tag{1.16}$$

From this we see that in our case the kinetic energy $K_{tot}$ is proportional to $\omega^2$ both before and after impact so that we get

$$K_f = r\ K_i = \left(\frac{11}{17}\right)^2 K_i = \frac{121}{289} K_i \tag{1.17}$$

so

$$\boldsymbol{r = 121/289} \approx 0.419 \tag{1.18}$$

c)

The kinetic energy $K_f$ after the impact must be sufficient to lift the center of mass to its highest position, straight above the point of contact. The angle through which $\vec{r}_C$ moves for this is

$$x = \frac{\alpha}{2} - \theta \tag{1.19}$$

where $\alpha = 60°$ is the top angle of the triangles meeting at the center of the polygon.[4] The energy for this lifting of the center of mass is

$$E_0 = Mga(1 - \cos x) = Mga\,(1 - \cos(30° - \theta)) \tag{1.20}$$

and we get the condition

$$K_f = rK_i > E_0 = Mga\,(1 - \cos(30° - \theta)) \tag{1.21}$$

thus

$$\boldsymbol{\delta = \frac{1}{r}\,(1 - \cos(30° - \theta))} \tag{1.22}$$

(Note that $\cos(30° - \theta) = \frac{\sqrt{3}}{2}\cos\theta + \frac{1}{2}\sin\theta$).

d)

Let $K_{i,n}$ and $K_{f,n}$ be the kinetic energies just before and just after the $n$th impact. We have shown that we have the relation

[4] In the general case $\alpha = 2\pi/N$.

$$K_{f,n} = r\ K_{i,n} \tag{1.23}$$

where $r = \frac{121}{289}$ for a hexagonal prism. Between subsequent impacts the height of the center of mass of the prism decreases by $a \sin\theta$ and its kinetic energy increases for this reason by

$$\Delta = Mga \sin\theta \tag{1.24}$$

We therefore have

$$K_{i,n+1} = rK_{i,n} + \Delta. \tag{1.25}$$

One does not have to write out the complete expression $K_{i,n}$ as a function of $K_{i,1}$ and $n$ to find the limit. This would actually be a proof that the limit exists (see below) but this is given in the problem text. Hence one can make $K_{i,n+1} \approx K_{i,n}$ arbitrarily accurate for sufficiently large $n$. The limit $K_{i,0}$ must thus satisfy the iterative formula, i.e.

$$K_{i,0} = rK_{i,0} + \Delta \tag{1.26}$$

yielding the solution

$$K_{i,0} = \frac{\Delta}{1-r}. \tag{1.27}$$

i.e.

$$\boldsymbol{\kappa = \frac{\sin\theta}{1-r}} \tag{1.28}$$

We can also solve the problem explicitly by writing out the full expressions:

$$\begin{aligned}
K_{i,2} &= r\ K_{i,1} + \Delta && (1.29)\\
K_{i,3} &= r\ K_{i,2} + \Delta = r^2 K_{i,1} + (1+r)\Delta && (1.30)\\
&\dots \\
K_{i,n} &= r^{n-1} K_{i,1} + (1 + r + \dots + r^{n-2})\Delta && (1.31)\\
&= r^{n-1} K_{i,1} + \frac{1-r^{n-1}}{1-r}\Delta && (1.32)
\end{aligned}$$

In the limit of $n \to \infty$ we get

$$K_{i,n} \to K_{i,0} = \frac{\Delta}{1-r} \tag{1.33}$$

which is, of course, the same result as before.

If we calculate the change in kinetic energy through a whole cycle, i.e. from just before impact number $n$ until just before impact $n+1$ we get

$$\begin{aligned}
\Delta K_{i,n} = K_{i,n+1} - K_{i,n} &= (r-1) r^{n-1} K_{i,1} + r^{n-1}\Delta && (1.34)\\
&= r^{n-1}\left(\Delta - (1-r)K_{i,1}\right) && (1.35)
\end{aligned}$$

This is positive if the initial value $K_{i,1} < K_{i,0}$ so that $K_{i,n}$ will then increase up to the limit value $K_{i,0}$. If, on the other hand, $K_{i,1} > K_{i,0}$, the kinetic energy $K_{i,n}$ just before impact will decrease down to the limit $K_{i,0}$.

All of this may remind you of motion with friction which increases with speed. Mathematically speaking, the main difference is that we here are dealing with difference equations instead of differential equations.

e)

For indefinite continuation the limit value of $K_i$ in part (d) must be larger than the minimum value for continuation found in part (c):

$$\frac{1}{1-r}\Delta = \frac{1}{1-r}Mga\sin\theta > Mga\left(1-\cos(30^\circ-\theta)\right)/r \tag{1.36}$$

We put $A = \frac{r}{1-r} = \frac{121}{168}$:

$$A\sin\theta > 1 - \cos 30^\circ\cos\theta - \sin 30^\circ\sin\theta \tag{1.37}$$

$$(A+1/2)\sin\theta + \sqrt{3}/2\cos\theta > 1 \tag{1.38}$$

To solve this we define[5]

$$u = \arccos\left(\frac{A+1/2}{\sqrt{(A+1/2)^2+3/4}}\right) \approx 35.36^\circ \tag{1.39}$$

and obtain

$$\cos u\sin\theta + \sin u\cos\theta > 1/\sqrt{(A+1/2)^2+3/4} \tag{1.40}$$

$$\sin(u+\theta) > 1/\sqrt{(A+1/2)^2+3/4} \tag{1.41}$$

$$\theta > \arcsin\{1/\sqrt{(A+1/2)^2+3/4}\} - u \approx 41.94^\circ - 35.36^\circ = 6.58^\circ \tag{1.42}$$

That is

$$\boldsymbol{\theta_0 \approx 6.58^\circ} \tag{1.43}$$

If $\theta > \theta_0$ *and* the kinetic energy before the first impact is sufficient according to part (c), we will, under the assumptions made, get an indefinite "rolling".

[5]You can of course solve any of the inequalities in a purely numerical way, e.g. by progressive guessing or by using the approximations $\sin\phi \approx \phi$ and $\cos\phi \approx 1 - \phi^2/2$.

## 1.3 Grading scheme

| Part **2(a)** | |
|---|---|
| Answer: $s = \omega_f/\omega_i = 11/17$, equation (1.12) | **3.5** |
| Part **2(b)** | |
| Answer: $r = K_f/K_i = s^2 = 121/289$, equation (1.18) | **1.0** |
| Part **2(c)** | |
| Answer: $K_{i,min}$ by $\delta$, equation (1.22) | **1.5** |
| Part **2(d)** | |
| Answer: Limit $K_{i,0}$ by $\kappa = \sin\theta/(1-r)$, equation (1.28) | **2.0** |
| Part **2(e)** | |
| Answer: Minimum angle $\theta_0 = 6.58°$, equation (1.43) | **2.0** |

# 2 Water under an ice cap[6]

## 2.1 Problem text

An ice cap is a thick sheet of ice (up to a few km in thickness) resting on the ground below and extending horizontally over tens or hundreds of km. In this problem we consider the melting of ice and the behavior of water under a temperate ice cap, i.e. an ice cap at the melting point. We may assume that under such conditions the ice causes pressure variations as a viscous fluid, but deforms in a brittle fashion, principally by vertical movement. For the purposes of this problem the following information is given.

| | |
|---|---|
| Density of water: | $\rho_w = 1.000 \cdot 10^3$ kg/m$^3$ |
| Density of ice: | $\rho_i = 0.917 \cdot 10^3$ kg/m$^3$ |
| Specific heat of ice: | $c_i = 2.1 \cdot 10^3$ J/(kg °C) |
| Specific latent heat of ice: | $L_i = 3.4 \cdot 10^5$ J/kg |
| Density of rock and magma: | $\rho_r = 2.9 \cdot 10^3$ kg/m$^3$ |
| Specific heat of rock and magma: | $c_r = 700$ J/(kg °C) |
| Specific latent heat of rock and magma: | $L_r = 4.2 \cdot 10^5$ J/kg |
| Average outward heat flow through the surface of the earth: | $J_Q = 0.06$ W/m$^2$ |
| Melting point of ice: | $T_0 = 0$°C, constant |

a) (0.5 points) Consider a thick ice cap at a location of average heat flow from the interior of the earth. Using the data from the table, calculate the thickness $d$ of the ice layer melted every year and write your answer in the designated box on the answer sheet.

b) (3.5 points) Consider now the upper surface of an ice cap. The ground below the ice cap has a slope angle $\alpha$. The upper surface of the cap slopes by an angle $\beta$ as shown in Figure 2.1. The vertical thickness of the ice at $x = 0$ is $h_0$. Hence the lower and upper surfaces of the ice cap can be described by the equations

$$y_1 = x \tan\alpha, \quad y_2 = h_0 + x \tan\beta \tag{2.1}$$

Derive an expression for the pressure $p$ at the bottom of the ice cap as a function of the horizontal coordinate $x$ and write it on the answer sheet.

Formulate mathematically a condition between $\beta$ and $\alpha$, so that water in a layer between the ice cap and the ground will flow in neither direction. Show that the condition is of the form $\tan\beta = s \tan\alpha$. Find the coefficient $s$ and write the result in a symbolic form on the answer sheet.

The line $y_1 = 0.8\ x$ in Figure 2.2 shows the surface of the earth below an ice cap. The vertical thickness $h_0$ at $x = 0$ is 2 km. Assume that water at the bottom is in equilibrium.

On a graph answer sheet draw the line $y_1$ and add a line $y_2$ showing the upper surface of the ice. Indicate on the figure which line is which.

[6]Authors: Gudni Axelsson and Thorsteinn Vilhjálmsson

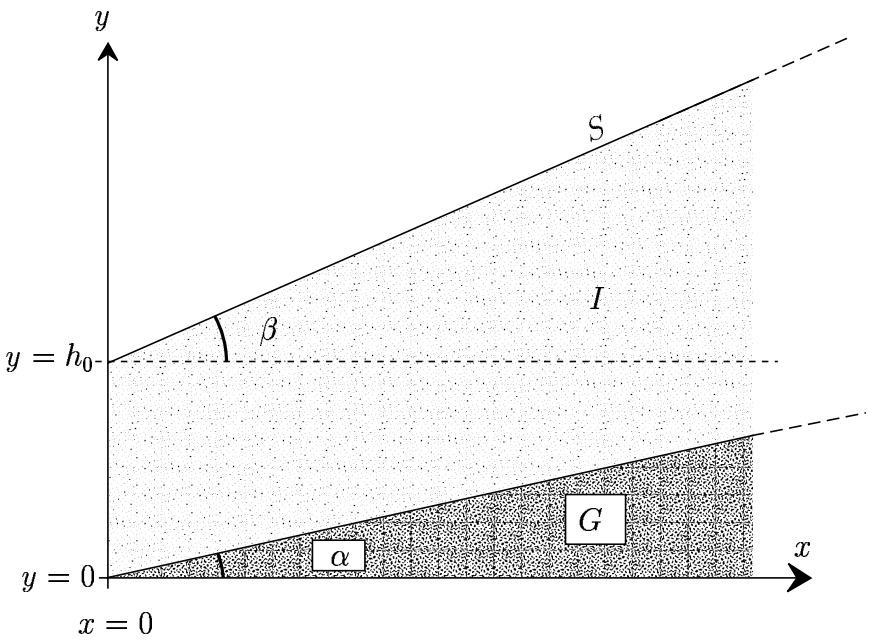


Figure 2.1: *Cross section of an ice cap with a plane surface resting on an inclined plane ground. S: surface, G: ground, I: ice cap.*

c) (1 point) Within a large ice sheet on horizontal ground and originally of constant thickness $D = 2.0$ km, a conical body of water of height $H = 1.0$ km and radius $r = 1.0$ km is formed rather suddenly by melting of the ice (Figure 2.3). We assume that the remaining ice adapts to this by vertical motion only.

Show analytically on a blank answer sheet and pictorially on a graph answer sheet, the shape of the surface of the ice cap after the water cone has formed and hydrostatic equilibrium has been reached.

d) (5 points) In its annual expedition an international group of scientists explores a temperate ice cap in Antarctica. The area is normally a wide plateau but this time they find a deep crater-like depression, formed like a top-down cone with a depth $h$ of 100 m and a radius $r$ of 500 m (Figure 2.4). The thickness of the ice in the area is 2000 m.

After a discussion the scientists conclude that most probably there was a minor volcanic eruption below the ice cap. A small amount of magma (molten rock) intruded at the bottom of the ice cap, solidified and cooled, melting a certain volume of ice. The scientists try as follows to estimate the volume of the intrusion and get an idea of what became of the melt water.

Assume that the ice only moved vertically. Also assume that the magma was completely molten and at 1200°C at the start. For simplicity, assume further that the intrusion had the form of a cone with a circular base vertically below the conical depression in the surface. The time for the rising of the magma was short relative to the time for the exchange of heat in the process. The heat flow is assumed to have been primarily vertical such that the volume melted from the ice at any time is bounded by a conical surface centered above the center of the magma intrusion.

Given these assumptions the melting of the ice takes place in two steps. At first the water is not in pressure equilibrium at the surface of the magma and hence flows away. The water flowing away can be assumed to have a temperature of 0°C. Subsequently,

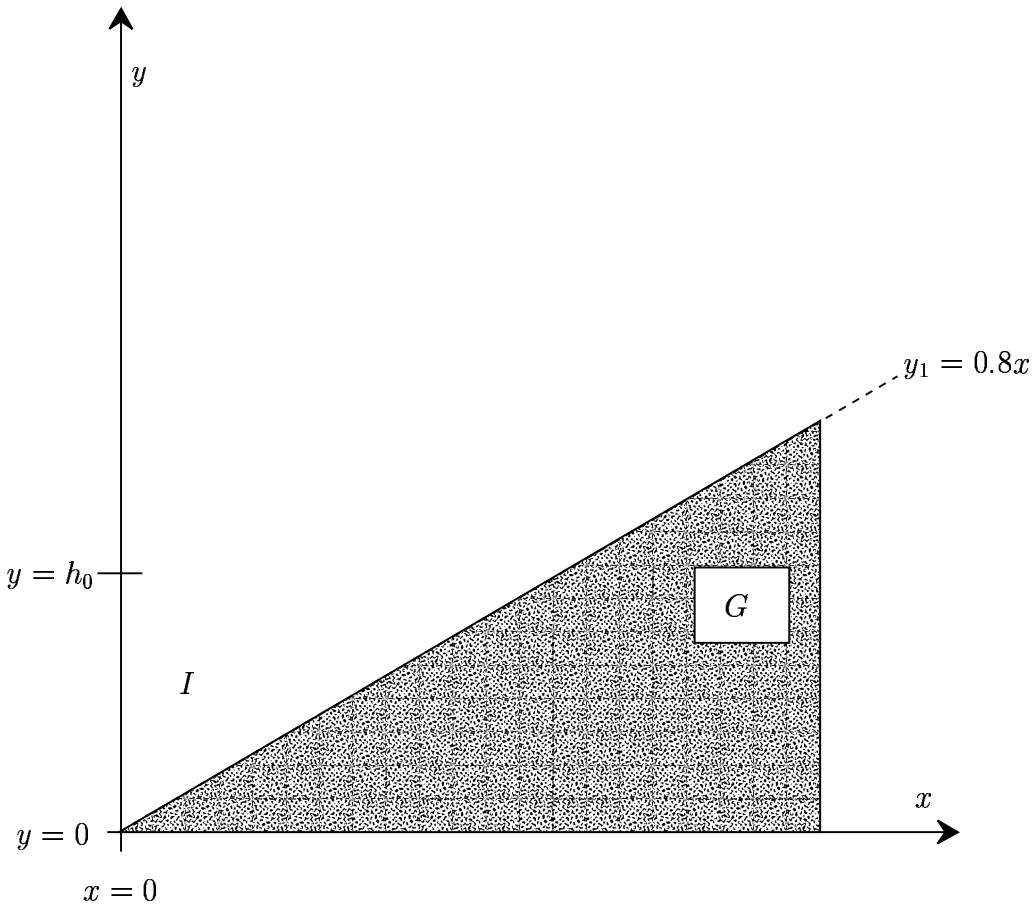


Figure 2.2: *Cross section of a temperate ice cap resting on an inclined ground with water at the bottom in equilibrium. G: ground, I: ice cap.*

hydrostatic equilibrium is reached and the water accumulates above the intrusion instead of flowing away.

When thermal equilibrium has been reached, you are asked to determine the following quantities. Write the answers on the answer sheet.

1. The height $H$ of the top of the water cone formed under the ice cap, relative to the original bottom of the ice cap.
2. The height $h_1$ of the intrusion.
3. The total mass $m_{tot}$ of the water produced and the mass $m'$ of water that flows away.

Plot on a graph answer sheet, to scale, the shapes of the rock intrusion and of the body of water remaining. Use the coordinate system suggested in Figure 2.4.

## 2.2 Solution

a)

Based on the conservation of energy we have

$$J_Q \cdot 1 \text{ year} = L_i \rho_i d \tag{2.2}$$

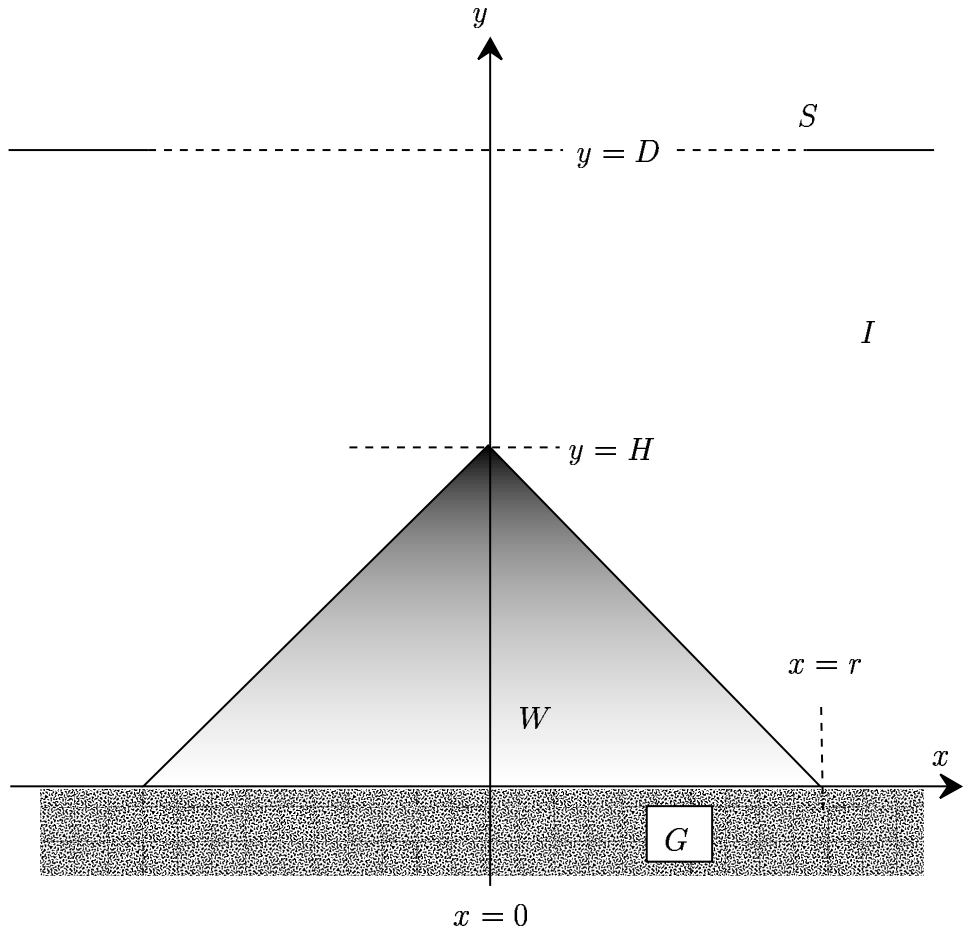


Figure 2.3: *A vertical section through the mid-plane of a water cone inside an ice cap. S: surface, W: water, G: ground, I: ice cap.*

$$\boldsymbol{d} = \frac{J_Q \cdot 1 \text{ year}}{L_i \ \rho_i} = \frac{0.06 \text{ J s}^{-1} \text{ m}^{-2} \ 365.25 \ \cdot 24 \ \cdot 60 \ \cdot 60 \text{ s}}{3.4 \cdot 10^5 \text{ J/kg } 917 \text{ kg/m}^3} = \mathbf{6.1 \cdot 10^{-3} m} \tag{2.3}$$

b)

Let $p_a$ be the atmospheric pressure, taken to be constant. At a depth $z$ inside the ice cap the pressure is given by:

$$p = \rho_i g z + p_a \tag{2.4}$$

Therefore, at the bottom of the ice cap, where $z = y_2 - y_1$:

$$\begin{aligned} \boldsymbol{p} &= \rho_i g(y_2 - y_1) + p_a & (2.5) \\ &= \boldsymbol{\rho_i g x(\tan\beta - \tan\alpha) + \rho_i g h_0 + p_a} & (2.6) \end{aligned}$$

For water not to move at the base of the ice cap the pressure must be hydrostatic (trivial, but can be seen from Bernoulli's equation), i.e.

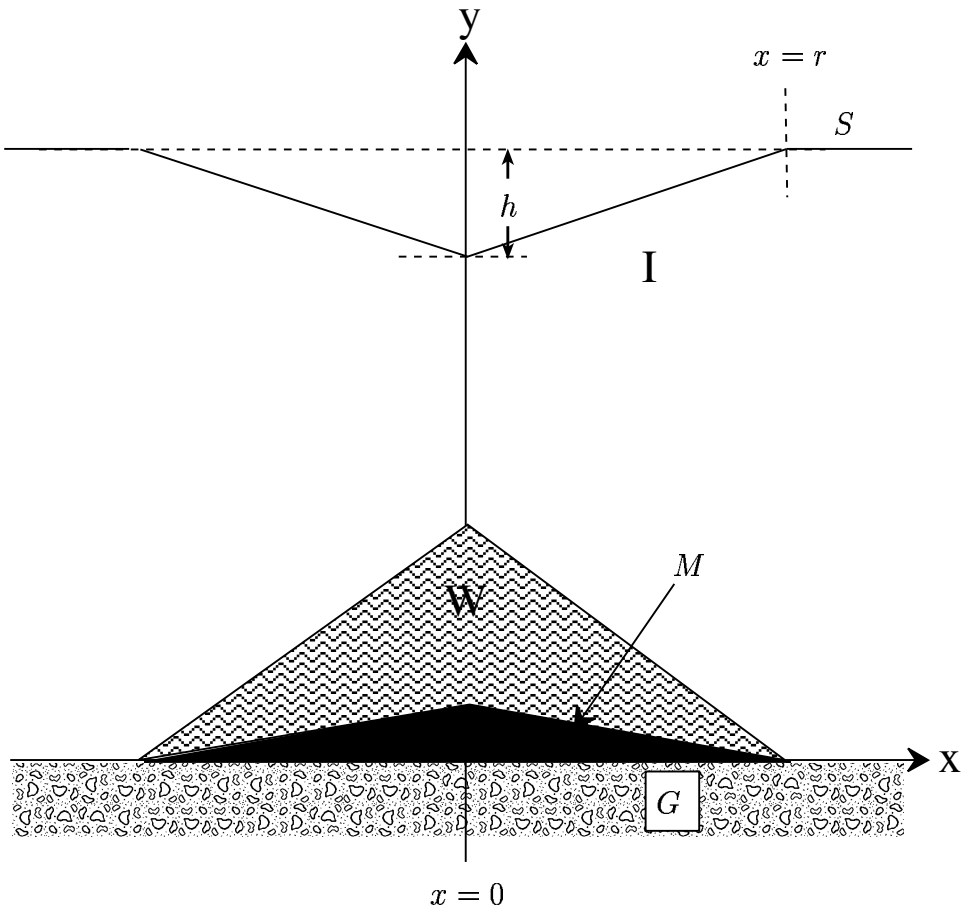


Figure 2.4: *A vertical and central cross section of a conical depression in a temperate ice cap. S: surface, G: ground, I: ice cap, M: rock/magma intrusion, W: water. Note that the figure is NOT drawn to scale.*

$$p = constant - \rho_w g y_1 \tag{2.7}$$
$$= constant - \rho_w g x \tan\alpha \tag{2.8}$$

Therefore

$$\rho_i g x(\tan\beta - \tan\alpha) = -\rho_w g x \tan\alpha \tag{2.9}$$

leading to

$$\boldsymbol{\tan\beta = -\frac{\rho_w - \rho_i}{\rho_i}\ \tan\alpha = -\frac{\Delta\rho}{\rho_i}\ \tan\alpha \approx -0.091\ \tan\alpha} \tag{2.10}$$
$$\boldsymbol{s = -\Delta\rho/\rho_i = -0.091} \tag{2.11}$$
$$\tag{2.12}$$

where the minus-sign is significant.

This can also be seen in various ways by looking at a mass element of water at the bottom of the ice and demanding equilibrium. – We now proceed with the solution.

With $\tan\alpha = 0.8$, we get $\tan\beta = -0.073$ and

$$\boldsymbol{y_2 = 2\ \mathrm{km} - 0.073\ x} \tag{2.13}$$

The students are supposed to draw this line on a graph.

c)

Since the ice adapts by vertical motion only we see that the conical depression at the surface will have the same radius of 1.0 km as the intrusion. According to (b) it will have a depth of

$$h = |r\tan\beta| = \frac{\Delta\rho}{\rho_i} r\tan\alpha \tag{2.14}$$

$$= \frac{\Delta\rho}{\rho_i} H \tag{2.15}$$

$$= 0.091\ \cdot 1\ \mathrm{km} = 91\ \mathrm{m}. \tag{2.16}$$

The students are supposed to show this result as a graph.

d)

The volume of a circular cone is $V = \frac{1}{3}\pi r^2 h$. We assume that the height of the intrusion is $h_1$. We may say that it firstly melts an ice cone of its own volume $V_1 = \frac{1}{3}\pi r^2 h_1$. Pressure equilibrium has not yet been reached. Hence the water will flow away and the ice will keep contact with the face of the intrusion making the upper surface of the ice horizontal again. The intrusion then melts a volume equivalent to a cone of height $h_2 = \frac{\Delta\rho}{\rho_i} h_1$ whereupon pressure equilibrium has been reached (following part (c)). During this second phase the melted water will also flow away. Assuming that the intrusion still has not cooled down to 0°C the intrusion will further melt a volume equivalent to a cone of height $h_3$, its water accumulating in place, forming a cone of height $h_3' = \frac{\rho_i}{\rho_w}\ h_3$ relative to the top of the intrusion. The total height of the ice cone melted is

$$h_{tot} = h_1 + h_2 + h_3 \tag{2.17}$$

The depth of the depression at the surface will be given by

$$h = \frac{\Delta\rho}{\rho_i}\ (h_1 + h_3') \tag{2.18}$$

which is most easily seen by considering pressure equilibrium in the final situation (again following part (c)). Thus, the requested height of the top of the water cone is

$$\boldsymbol{H} = h_1 + h_3' = \frac{\rho_i}{\Delta\rho}\ h = \mathbf{1.1 \times 10^3 m} \tag{2.19}$$

The heat balance gives

$$\frac{1}{3}\ \pi\ r^2\ \{\rho_r h_1 (L_r\ + c_r\ \Delta T) - \rho_i\ L_i\ h_{tot}\} = 0 \tag{2.20}$$

where $\Delta T = 1200°\text{C}$ is the change in temperature of the rock intrusion. Following equation (2.17) and using the facts that $h_2 = \frac{\Delta\rho}{\rho_i} h_1$ and $h_3 = \frac{\rho_w}{\rho_i} h_3'$ we obtain

$$h_{tot} = h_1 + \frac{\Delta\rho}{\rho_i} h_1 + \frac{\rho_w}{\rho_i} h_3' = \frac{\rho_w}{\rho_i} (h_1 + h_3') \tag{2.21}$$

Therefore (using equation (2.19))

$$h_{tot} = \frac{\rho_w}{\rho_i} (h_1 + h_3') = \frac{\rho_w}{\rho_i} H = \frac{\rho_w}{\Delta\rho} h = 1.20 \cdot 10^3 \text{m} \tag{2.22}$$

This implies that the cone does not reach the surface of the ice cap. Inserting the result into the equation (2.20) we can solve for $h_1$:

$$\rho_r \; h_1 \; (L_r + \; c_r \; \Delta T) = \frac{\rho_i \; \rho_w \; L_i \; h}{\Delta\rho} \tag{2.23}$$

$$\begin{aligned} \boldsymbol{h_1} &= \frac{\rho_i \; \rho_w \; L_i \; h}{\Delta\rho \; \rho_r \; (L_r + \; c_r \; \Delta T)} && (2.24) \\ &= \mathbf{103 \text{ m}} && (2.25) \end{aligned}$$

The total mass of water formed is of course equal to the mass of the ice melted and is

$$\boldsymbol{m_{tot}} = \rho_i \; (1/3) \; \pi \; r^2 \; h_{tot} = \mathbf{2.9 \cdot 10^{11} \text{ kg}} \tag{2.26}$$

The mass of the water which flows away is

$$\boldsymbol{m'} = \frac{h_1 + h_2}{h_{tot}} \; m_{tot} = \frac{\rho_w \; h_1}{\rho_i \; h_{tot}} \; m_{tot} = \mathbf{2.7 \cdot 10^{10} \text{ kg}} \tag{2.27}$$

The students are finally expected to plot the shapes of the rock intrusion and the water body.

## 2.3 Grading scheme

| **2(a)** | |
|---|---|
| Answer: equation (2.3), $d = 6.1 \cdot 10^{-3}$ m | **0.5** |
| **2(b)** | |
| Answer i): equation (2.6): $p = \rho_i g x(\tan\beta - \tan\alpha) + \rho_i g h_0 + p_a$ | **1.0** |
| Answer ii): equation (2.10): $s = -\frac{\rho_w - \rho_i}{\rho_i} = -\frac{\Delta\rho}{\rho_i}$ | **2.0** |
| Answer iii): Graph based on equation (2.13) | **0.5** |
| **2(c)** | |
| Answer: Depth, radius and graph, $r = 1000$ m, $h = 91$ m | **1.0** |
| **2(d)** | |
| Answer i): Height of water cone as in (2.19): $H = 1.1 \cdot 10^3$ m | **2.0** |
| Answer ii): Height of intrusion as in (2.25): $h_1 = 103$ m | **1.0** |
| Answer iii): Total mass of melt water as in (2.26): $m_{tot} = 2.9 \cdot 10^{11}$ kg | **0.5** |
| Answer iv): Mass of water flowing away as in (2.27): $m' = 2.7 \cdot 10^{10}$ kg | **1.0** |
| Answer v): Graph | **0.5** |

# 3 Faster than light?[7]

## 3.1 Problem text

In this problem we analyze and interpret measurements made in 1994 on radio wave emission from a compound source within our galaxy.

The receiver was tuned to a broad band of radio waves of wavelengths of several centimeters. Figure 3.1 shows a series of images recorded at different times. The contours indicate constant radiation strength in much the same way as altitude contours on a geographical map. In the figure the two maxima are interpreted as showing two objects moving away from a common center shown by crosses in the images. (The center, which is assumed to be fixed in space, is also a strong radiation emitter but mainly at other wavelengths). The measurements conducted on the various dates were made at the same time of day.

The scale of the figure is given by a line segment showing one arc second (as). (1 as = 1/3600 of a degree). The distance to the celestial body at the center of the figure, indicated by crosses, is estimated to be $R = 12.5$ kpc. A kiloparsec (kpc) equals $3.09 \cdot 10^{19}$ m. The speed of light is c = $3.00 \cdot 10^8$ m/s. Error calculations are not required in the solution.

a) (2 points) We denote the angular positions of the two ejected radio emitters, relative to the common center, by $\theta_1(t)$ and $\theta_2(t)$, where the subscripts 1 and 2 refer to the left and right hand ones, respectively, and $t$ is the time of observation. The angular speeds, as seen from the Earth, are $\omega_1$ and $\omega_2$. The corresponding apparent transverse linear speeds of the two sources are denoted by $v'_{1,\perp}$ and $v'_{2,\perp}$.

Using Figure 3.1, make a graph to find the numerical values of $\omega_1$ and $\omega_2$ in milli-arc-seconds per day (mas/d). Also determine the numerical values of $v'_{1,\perp}$ and $v'_{2,\perp}$, and write all answers on the answer sheet. (You may be puzzled by some of the results).

b) (3 points) In order to resolve the puzzle arising in part (a), consider a light-source moving with velocity $\vec{v}$ at an angle $\phi$ ($0 \leq \phi \leq \pi$) to the direction towards a distant observer $O$ (Figure 3.2). The speed may be written as $v = \beta c$, where $c$ is the speed of light. The distance to the source, as measured by the observer, is $R$. The angular speed of the source, as seen from the observer, is $\omega$, and the apparent linear speed perpendicular to the line of sight is $v'_\perp$.

Find $\omega$ and $v'_\perp$ in terms of $\beta$, $R$ and $\phi$ and write your answer on the answer sheet.

c) (1 point) We assume that the two ejected objects, described in the introduction and in part (a), are moving in opposite directions with equal speeds $v = \beta c$. Then the results of part (b) make it possible to calculate $\beta$ and $\phi$ from the angular speeds $\omega_1$ and $\omega_2$ and the distance $R$. Here $\phi$ is the angle defined in part (b), for the left hand object, corresponding to subscript 1 in part (a).

Derive formulas for $\beta$ and $\phi$ in terms of known quantities and determine their numerical values from the data in part (a). Write your answers in the designated fields on the answer sheet.

d) (2 points) In the one-body situation of part (b), find the condition for the apparent perpendicular speed $v'_\perp$ to be larger than the speed of light $c$.

[7]Authors: Einar Gudmundsson, Knútur Árnason and Thorsteinn Vilhjálmsson

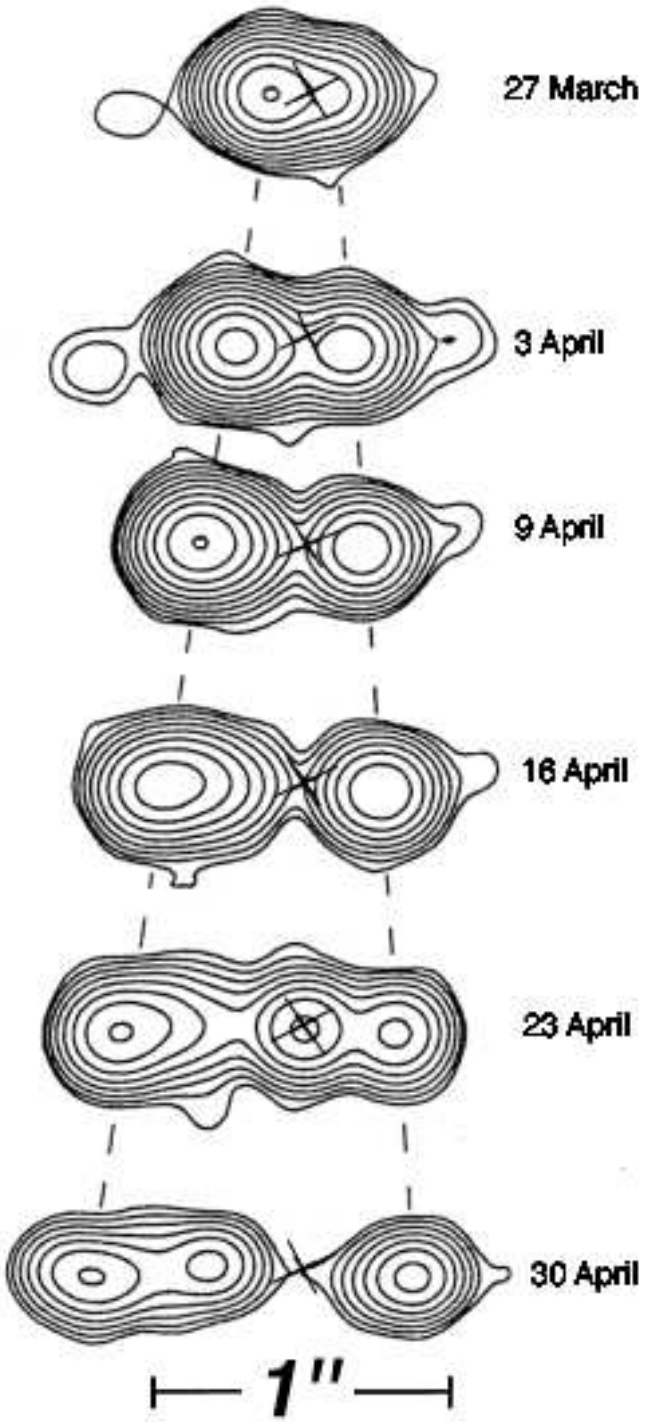


Figure 3.1: *Radio emission from a source in our galaxy.*

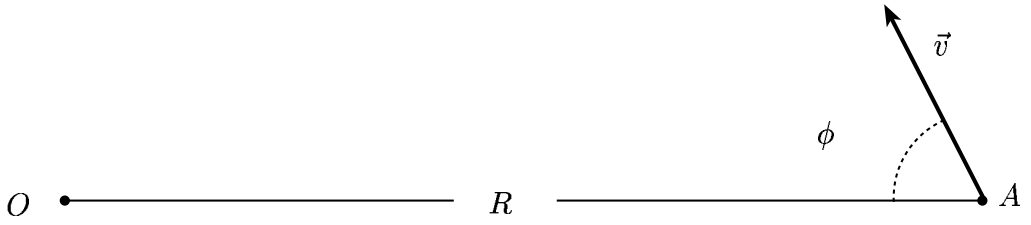


Figure 3.2: *The observer is at $O$ and the original position of the light source is at $A$. The velocity vector is $\vec{v}$.*

Write the condition in the form $\beta > f(\phi)$ and provide an analytic expression for the function $f$ on the answer sheet.

Draw on the graph answer sheet the physically relevant region of the $(\beta, \phi)$-plane. Show by shading in which part of this region the condition $v'_\perp > c$ holds.

e) (1 point) Still in the one-body situation of part (b), find an expression for the maximum value $(v'_\perp)_{max}$ of the apparent perpendicular speed $v'_\perp$ for a given $\beta$ and write it in the designated field on the answer sheet. Note that this speed increases without limit when $\beta \to 1$.

f) (1 point) The estimate for $R$ given in the introduction is not very reliable. Scientists have therefore started speculating on a better and more direct method for determining $R$. One idea for this goes as follows. Assume that we can identify and measure the Doppler shifted wavelengths $\lambda_1$ and $\lambda_2$ of radiation from the two ejected objects, corresponding to the same known original wavelength $\lambda_0$ in the rest frames of the objects.

Starting from the equations for the relativistic Doppler shift,
$\lambda = \lambda_0(1 - \beta\cos\phi)(1-\beta^2)^{-1/2}$, and assuming, as before, that both objects have the same speed, $v$, show that the unknown $\beta = v/c$ can be expressed in terms of $\lambda_0$, $\lambda_1$, and $\lambda_2$ as

$$\beta = \sqrt{1 - \frac{\alpha\ \lambda_0^2}{(\lambda_1 + \lambda_2)^2}}\ . \tag{3.1}$$

Write the numerical value of the coefficient $\alpha$ in the designated field on the answer sheet.

You may note that this means that the suggested wavelength measurements will in practice provide a new estimate of the distance.

## 3.2 Solution

a) On Figure 3.1 we mark the centers of the sources as neatly as we can. Let $\theta_1(t)$ be the angular distance of the left center from the cross as a function of time and $\theta_2(t)$ the angular distance of the right center. We measure these quantities on the figure at the given times by a ruler and convert to arcseconds according to the given scale. This results in the following numerical data:

| time [days] | $\theta_1$ [as] | $\theta_2$ [as] |
|---|---|---|
| 0 | 0.139 | 0.076 |
| 7 | 0.253 | 0.139 |
| 13 | 0.354 | 0.190 |
| 20 | 0.468 | 0.253 |
| 27 | 0.601 | 0.316 |
| 34 | 0.709 | 0.367 |

The uncertainty in the readings by the ruler is estimated to be $\pm 0.5$ mm, resulting in the uncertainty of $\pm$ 0.013 as in the $\theta$ values. We plot the data in Figure 3.3.

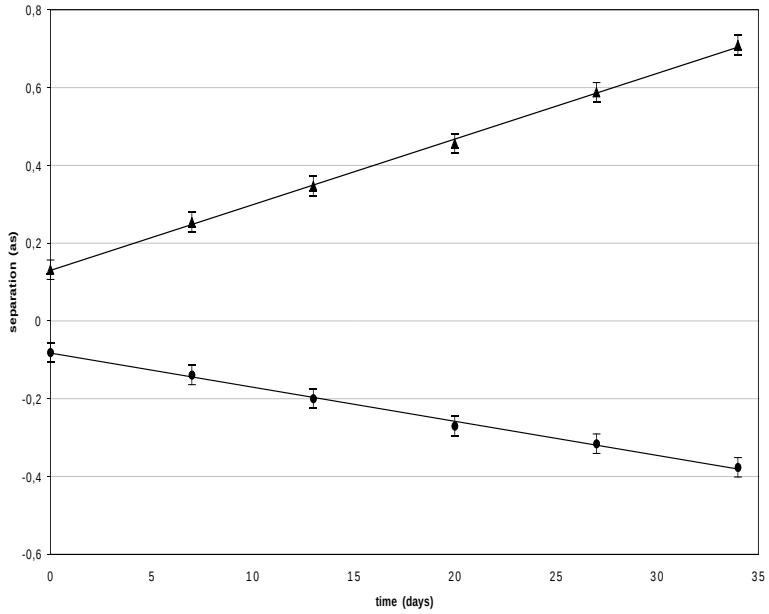


Figure 3.3: *The angular distances $\theta_1$ and $\theta_2$ (in as) as functions of the time in days.*

Fitting straight lines through the data results in:

$$\boldsymbol{\omega_1} = \mathrm{d}\theta_1/\mathrm{d}t = \mathbf{(17.0 \pm 1.0)\ mas/day} = 9.54 \cdot 10^{-13}\mathrm{rad/s} \quad (3.2)$$

$$\boldsymbol{\omega_2} = \mathrm{d}\theta_2/\mathrm{d}t = \mathbf{(8.7 \pm 1.0)\ mas/day} = 4.88 \cdot 10^{-13}\mathrm{rad/s} \quad (3.3)$$

$$\boldsymbol{v'_{1,\perp}} = \omega_1\ R = 9.54 \cdot 10^{-13} \cdot 12.5 \cdot 3.09 \cdot 10^{19} \quad (3.4)$$

$$= \mathbf{3.68 \cdot 10^8 m/s \approx (1.23 \pm 0.07)\ \boldsymbol{c}} \quad (3.5)$$

$$\boldsymbol{v'_{2,\perp}} = \mathbf{1.89 \cdot 10^8 m/s \approx (0.63 \pm 0.07)\ \boldsymbol{c}} \quad (3.6)$$

b) We consider the motion of the source during the time interval $\Delta t$ from the point $A$ to the point $A'$, see Figure 3.4.

We then have

$$\vec{r}_{AA'} = \vec{r}_{A'} - \vec{r}_A = \vec{v} \cdot \Delta t\ . \quad (3.7)$$

Now let $\Delta t'$ denote the difference in arrival times at $O$ of the signals from $A$ and $A'$. Due to the different distances to $A$ and $A'$ and the finite speed of light, c, we have

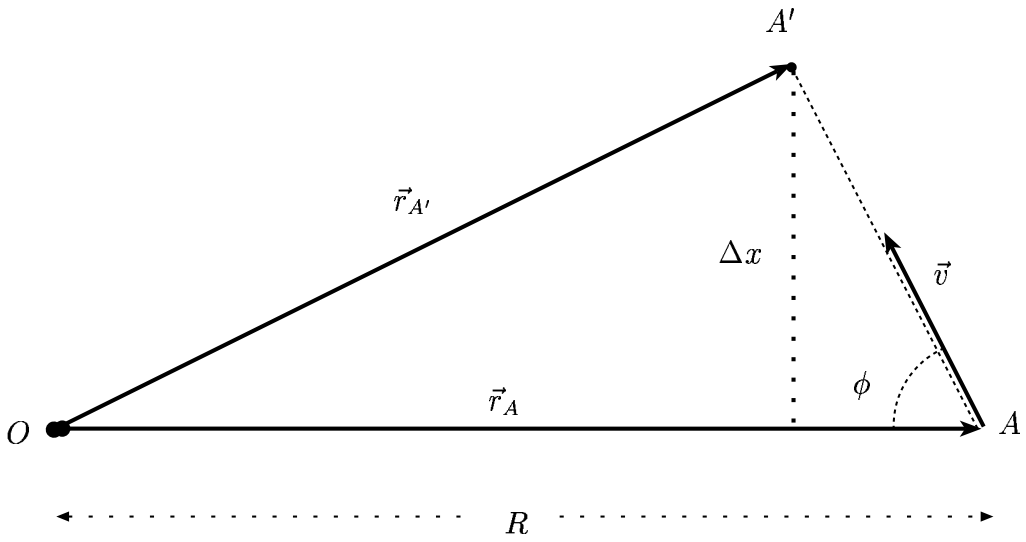


Figure 3.4: *The observer is at $O$ and the original position of the source is at $A$. The velocity vector is $\vec{v}$.*

$$\Delta t' = \Delta t + (r_{A'} - r_A)/c \ . \tag{3.8}$$

For small $\Delta t$, such that $v\ \Delta t << r_A = R$, we have

$$r_{A'} - r_A \approx -v\ \Delta t \cos\phi \tag{3.9}$$

and hence

$$\Delta t' \approx \Delta t\ (1 - \beta\cos\phi)\ ;\ \beta = v/c \ . \tag{3.10}$$

This implies that an observer at $O$ will find the apparent transverse speed of the source to be

$$\boldsymbol{v'_\perp = \frac{\Delta x}{\Delta t'} = \frac{\Delta x}{\Delta t\ (1 - \beta\cos\phi)} = \frac{c\beta\sin\phi}{1 - \beta\cos\phi}} \tag{3.11}$$

where we have used that the real transverse speed in the reference frame of the observer is $v_\perp = \Delta x/\Delta t = c\beta\sin\phi$.

The angular speed observed at $O$ is

$$\boldsymbol{\omega = \frac{v'_\perp}{R} = \frac{c\beta\sin\phi}{R\ (1 - \beta\cos\phi)}} \tag{3.12}$$

c) Figure 3.5 shows the situation in this case. Note the relations given in the caption. Taking $\phi = \phi_1$ we have $\sin\phi_2 = \sin\phi$ and $\cos\phi_2 = -\cos\phi$. Equation (3.12) then gives:

$$\omega_1 = \frac{\beta\ c\ \sin\phi}{R\ (1 - \beta\cos\phi)} \tag{3.13}$$

$$\omega_2 = \frac{\beta\ c\ \sin\phi}{R\ (1 + \beta\cos\phi)} \ . \tag{3.14}$$

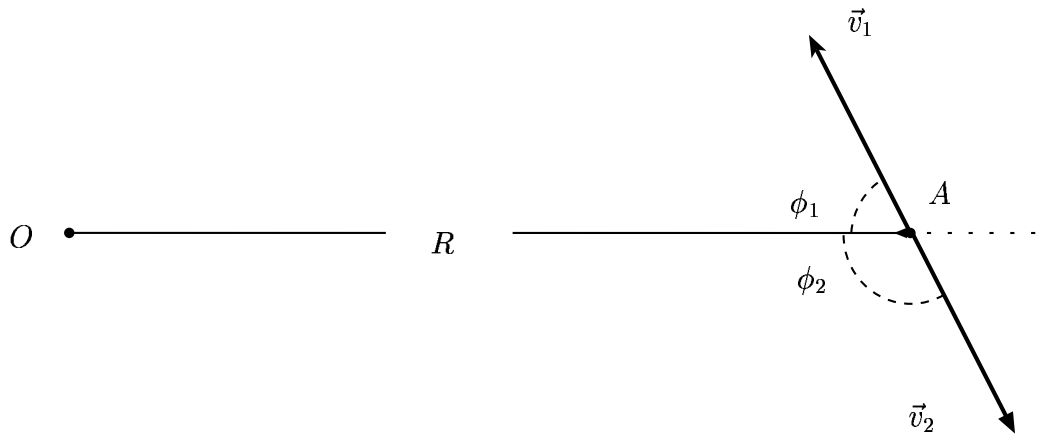


Figure 3.5: *If the two objects have equal speeds but opposite velocities we have* $v_1 = v_2 = v$, $\beta_1 = \beta_2 = \beta$ *and* $\phi_2 = \pi - \phi_1$.

The quantities $\omega_1$, $\omega_2$ and $R$ are given, but $\beta$ and $\phi$ are to be determined as stated in the problem text. Simple algebra gives:

$$(1 - \beta \cos\phi)\, \omega_1\, \omega_2 = \beta\, c\, \sin\phi\, \omega_2/R \tag{3.15}$$

$$(1 + \beta \cos\phi)\, \omega_2\, \omega_1 = \beta\, c\, \sin\phi\, \omega_1/R\,. \tag{3.16}$$

Subtracting (3.15) from (3.16) gives:

$$2\, \beta \cos\phi\, \omega_2\, \omega_1 = \beta\, c\, \sin\phi\, (\omega_1 - \omega_2)/R \tag{3.17}$$

$$\tan\phi = \frac{2\, R\, \omega_2\, \omega_1}{c\ (\omega_1 - \omega_2)} \tag{3.18}$$

$$\boldsymbol{\phi = \arctan\left(\frac{2\, R\, \omega_2\, \omega_1}{c\ (\omega_1 - \omega_2)}\right)}\,. \tag{3.19}$$

Dividing (3.15) by (3.16) gives $\beta$ in terms of $\cos\phi$ and the known quantities $\omega_1$ and $\omega_2$:

$$\omega_1\, (1 - \beta \cos\phi) = \omega_2\, (1 + \beta \cos\phi) \tag{3.20}$$

$$\boldsymbol{\beta = \frac{\omega_1 - \omega_2}{\cos\phi\ (\omega_1 + \omega_2)}}\,. \tag{3.21}$$

Inserting the values of $\omega_1$ and $\omega_2$ from part (a) and the given values of $R$ and $c$ we get:

$$\boldsymbol{\phi} = \arctan(2.57) = \mathbf{1.20\ rad = 68.8^\circ \pm 2^\circ} \tag{3.22}$$

$$\boldsymbol{\beta} = \mathbf{0.892 \pm 0.08} \tag{3.23}$$

d) Equation (3.11) shows that the observer will find the apparent transverse speed to be larger than or equal to the speed of light if and only if:

$$\frac{\beta \sin\phi}{1-\beta\cos\phi} \geq 1. \tag{3.24}$$

If $\beta < 1$ condition (3.24) is equivalent to:

$$\begin{aligned} \beta\sin\phi &\geq 1-\beta\cos\phi && (3.25)\\ \beta\left(\sin\phi+\cos\phi\right) &\geq 1 && (3.26)\\ \beta\ \sqrt{2}\left(\sin\phi\cos\frac{\pi}{4}+\cos\phi\sin\frac{\pi}{4}\right) &\geq 1 && (3.27)\\ \sin\left(\phi+\frac{\pi}{4}\right) &\geq \frac{1}{\beta\sqrt{2}} && (3.28) \end{aligned}$$

and hence (3.24) is satisfied if:

$$\beta > \boldsymbol{f(\phi) = \left(\sqrt{2}\,\sin(\phi+\pi/4)\right)^{-1}}\ . \tag{3.29}$$

The physically relevant region in the $(\beta,\phi)$-plane is:

$$\boldsymbol{(\beta,\phi) \in [0,1[\times[0,\pi]}\ . \tag{3.30}$$

It is obvious that (3.24) can only be satisfied for $\phi \in [0,\pi/2]$ and (3.28) can only have a solution for $\phi$ if $\beta \geq 1/\sqrt{2}$.

We therefore take a closer look at the region

$$(\beta,\phi)\ \in\ [2^{-1/2},1[\ \times\ [0,\pi/2] \tag{3.31}$$

The mapping

$$(\beta,\phi) \mapsto \beta\sin\left(\phi+\frac{\pi}{4}\right) \tag{3.32}$$

is continuous in this region. It is therefor sufficient to look at the boundary of the region, defined by the equality sign in (3.28):

$$\beta\sin\left(\phi+\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} \tag{3.33}$$

This defines $\beta$ as a function of $\phi$ which is shown in Figure 3.6 as the curve bounding the shaded area where $v'_\perp > c$.

e) To find the extrema of $v'_\perp$ as a function of $\phi$ we differentiate (3.11) and get

$$\frac{d}{d\phi}\left(\frac{v'_\perp}{c}\right) = \frac{\beta(\cos\phi-\beta)}{(1-\beta\cos\phi)^2}\ . \tag{3.34}$$

This is zero for $\phi = \phi_m$ where:

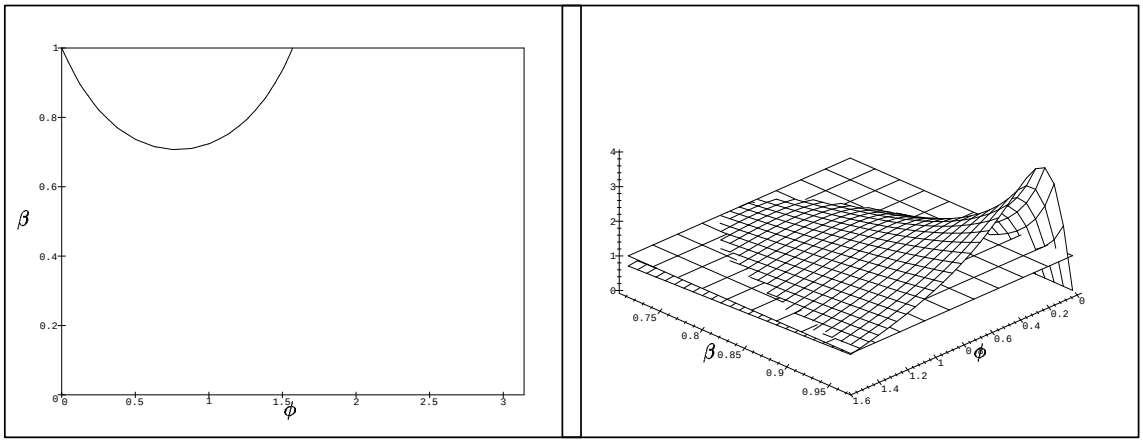


Figure 3.6: *The region between the horizontal line and the curve in the upper left hand corner shows where* $v'_\perp/c > 1$.

Figure 3.7: *The curved surface is* $v'_\perp/c$ *as a function of* $\beta$ *and* $\phi$. *The plane represents the constant function* $\beta = 1$.

$$\cos\phi_m = \beta \ ; \ \phi_m = \arccos\beta \in ]0, \pi/2] \tag{3.35}$$

To see that this is indeed a maximum, we differentiate (3.34) again and get:

$$\frac{d^2}{d\phi^2}\left(\frac{v'_\perp}{c}\right) = -\beta\left(\frac{\sin\phi}{(1-\beta\cos\phi)^2} + 2\frac{\beta\sin\phi(\cos\phi-\beta)}{(1-\beta\cos\phi)^3}\right) \tag{3.36}$$

At the extremum

$$\left.\frac{d^2}{d\phi^2}\left(\frac{v'_\perp}{c}\right)\right|_{\phi_m} = -\frac{\beta\sin\phi_m}{(1-\beta^2)^2} < 0 \tag{3.37}$$

showing that $\phi_m$ corresponds to a maximum. From (3.11) and (3.35) the maximum apparent transverse speed is given:

$$\boldsymbol{(v'_\perp)_{max} = \frac{\beta c}{\sqrt{1-\beta^2}}} \tag{3.38}$$

From this and (3.35) we see that

$$(v'_\perp)_{max} \underset{\beta\to 1}{\longrightarrow} \infty \ ; \ \phi_m \underset{\beta\to 1}{\longrightarrow} 0 \ . \tag{3.39}$$

Figure 3.7 shows $v'_\perp/c$ as a function of $\beta$ and $\phi$ in the region $(\beta, \phi) \in [2^{-1/2}, 1[ \times [0, \pi/2]$.

f) We have the equations for relativistic Doppler-shift:

$$\frac{\lambda_{1,2}}{\lambda_0} = \frac{1 \mp \beta \cos\phi}{\sqrt{1-\beta^2}} \tag{3.40}$$

We add them, define an auxiliary ratio $\rho$ and solve for $\beta$.

$$\rho := \frac{\lambda_1 + \lambda_2}{2\ \lambda_0} = \frac{1}{\sqrt{1-\beta^2}} \tag{3.41}$$

$$\rho^2\ (1-\beta^2) = 1 \tag{3.42}$$

$$\beta = \sqrt{1 - 1/\rho^2} = \sqrt{1 - \frac{4\ \lambda_0^2}{(\lambda_1+\lambda_2)^2}} \tag{3.43}$$

giving

$$\boldsymbol{\alpha = 4} \tag{3.44}$$

Adding equation (3.43) to the set of equations (3.18) and (3.21) we have three equations which can be solved for the three unknowns $\beta$, $\phi$ and $R$. For instance, we may calculate $\beta$ from (3.43), insert that into (3.21), and solve for $\phi$. The distance $R$ can then be obtained from (3.18). Thus the measurement of the Doppler-shifted wavelengths turns out to give an estimate of the distance to the source provided that $\omega_1$ and $\omega_2$ are known.

## 3.3 Grading scheme

| **Part 1(a)** | |
|---|---|
| Answer i): equation (3.2), $\omega_1$ in the range (16.5-17.5) mas/day | **0.8** |
| Answer ii): equation (3.3), $\omega_2$ in the range (8.2-9.2) mas/day | **0.8** |
| Answer iii): equation (3.4), for $v'_{1,\perp}$ in the range (1.13-1.30)c | **0.2** |
| Answer iv): equation (3.6), for $v'_{2,\perp}$ in the range (0.56-0.70)c | **0.2** |
| **Part 1(b)** | |
| Answer i): $v'_\perp(\beta,\phi)$, equation (3.11) | **2.5** |
| Answer ii): $\omega(\beta,\phi)$, equation (3.12) | **0.5** |
| **Part 1(c)** | |
| Answer i): $\phi(\omega_1,\omega_2)$, equation (3.19) | **0.3** |
| Answer ii): $\beta(\omega_1,\omega_2)$, equation (3.21) | **0.3** |
| Answer iii): $\phi$ numerical in the range 67° - 71° | **0.2** |
| Answer iv): $\beta$ numerical in the range 0.81-0.97 | **0.2** |
| **Part 1(d)** | |
| Answer i): Condition $\beta > f(\phi)$, equation (3.29) | **1.0** |
| Answer ii): Condition on $(\beta,\phi)$, graph | **1.0** |
| **Part 1(e)** | |
| Answer: $(v'_\perp)_{max}$, equation (3.38) | **1.0** |
| **Part 1(f)** | |
| Answer: $\beta$ in terms of $\lambda$-s, by $\alpha$, equation (3.44) | **1.0** |