

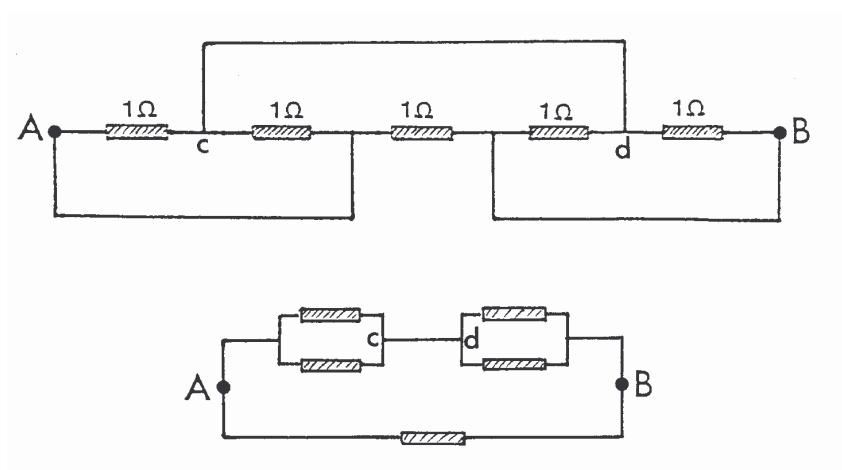


# 27<sup>th</sup> INTERNATIONAL PHYSICS OLYMPIAD OSLO, NORWAY

## THEORETICAL COMPETITION JULY 2 1996

### Solution Problem 1

a) The system of resistances can be redrawn as shown in the figure:



The equivalent drawing of the circuit shows that the resistance between point c and point A is  $0.5\Omega$ , and the same between point d and point B. The resistance between points A and B thus consists of two connections in parallel: the direct  $1\Omega$  connection and a connection consisting of two  $0.5\Omega$  resistances in series, in other words two parallel  $1\Omega$  connections. This yields

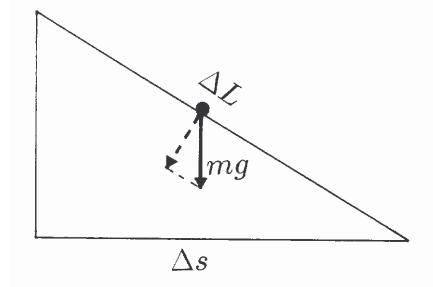
$$R = \underline{\underline{0.5\Omega}}.$$

b) For a sufficiently short horizontal displacement  $\Delta s$  the path can be considered straight. If the corresponding length of the path element is  $\Delta L$ , the friction force is given by

$$\mu mg \frac{\Delta s}{\Delta L}$$

and the work done by the friction force equals force times displacement:

$$\mu mg \frac{\Delta s}{\Delta L} \cdot \Delta L = \mu mg \Delta s.$$



Adding up, we find that along the whole path the total work done by friction forces is  $\mu mg s$ . By energy conservation this must equal the decrease  $mg h$  in potential energy of the skier. Hence

$$h = \underline{\underline{\mu s.}}$$


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c) Let the temperature increase in a small time interval  $dt$  be  $dT$ . During this time interval the metal receives an energy  $P dt$ .

The heat capacity is the ratio between the energy supplied and the temperature increase:

$$C_p = \frac{P dt}{dT} = \frac{P}{dT/dt}.$$

The experimental results correspond to

$$\frac{dT}{dt} = \frac{T_0}{4} a[1 + a(t - t_0)]^{-3/4} = T_0 \frac{a}{4} \left( \frac{T_0}{T} \right)^3.$$

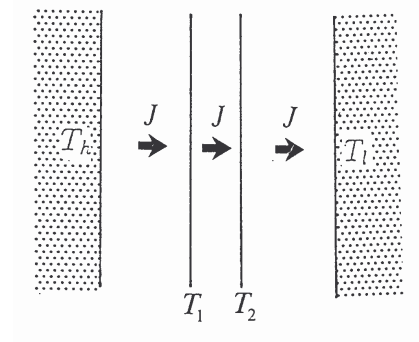
Hence

$$C_p = \frac{P}{dT/dt} = \frac{4P}{\underline{\underline{aT_0^4}}} T^3.$$

(*Comment:* At low, but not extremely low, temperatures heat capacities of metals follow such a  $T^3$  law.)

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d)



Under stationary conditions the net heat flow is the same everywhere:

$$J = \sigma(T_h^4 - T_1^4)$$

$$J = \sigma(T_1^4 - T_2^4)$$

$$J = \sigma(T_2^4 - T_l^4)$$

Adding these three equations we get

$$3J = \sigma(T_h^4 - T_l^4) = J_0,$$

where  $J_0$  is the heat flow in the absence of the heat shield. Thus  $\xi = J/J_0$  takes the value

$$\underline{\underline{\xi = \frac{1}{3}}}.$$

e) The magnetic field can be determined as the superposition of the fields of two *cylindrical* conductors, since the effects of the currents in the area of intersection cancel. Each of the cylindrical conductors must carry a larger current  $I'$ , determined so that the fraction  $I$  of it is carried by the actual cross section (the moon-shaped area).

The ratio between the currents  $I$  and  $I'$  equals the ratio between the cross section areas:

$$\frac{I}{I'} = \frac{(\frac{\pi}{12} + \frac{\sqrt{3}}{8})D^2}{\frac{\pi}{4}D^2} = \frac{2\pi + 3\sqrt{3}}{6\pi}.$$

*Inside* one cylindrical conductor carrying a current  $I'$  Ampère's law yields at a distance  $r$  from the axis an azimuthal field

$$B_\phi = \frac{\mu_0}{2\pi r} \frac{I' \pi r^2}{\frac{\pi}{4} D^2} = \frac{2\mu_0 I' r}{\pi D^2}.$$

The cartesian components of this are

$$B_x = -B_\phi \frac{y}{r} = -\frac{2\mu_0 I' y}{\pi D^2}; \quad B_y = B_\phi \frac{x}{r} = \frac{2\mu_0 I' x}{\pi D^2}.$$

For the superposed fields, the currents are  $\pm I'$  and the corresponding cylinder axes are located at  $x = \mp D/4$ .

The two x-components add up to zero, while the y-components yield

$$B_y = \frac{2\mu_0}{\pi D^2} [I'(x + D/4) - I'(x - D/4)] = \frac{\mu_0 I'}{\pi D} = \frac{6\mu_0 I}{\underline{\underline{(2\pi + 3\sqrt{3})D}}},$$

i.e., a *constant* field. The direction is along the positive y-axis.

## Solution problem 2

a) The potential energy gain  $eV$  is converted into kinetic energy. Thus

$$\frac{1}{2}mv^2 = eV \quad (\text{non-relativistically})$$

$$\frac{mc^2}{\sqrt{1-v^2/c^2}} - mc^2 = eV \quad (\text{relativistically}).$$

Hence

$$v = \begin{cases} \sqrt{2eV/m} & (\text{non - relativistically}) \\ c\sqrt{1 - \left(\frac{mc^2}{mc^2 + eV}\right)^2} & (\text{relativistically}). \end{cases} \quad (1)$$

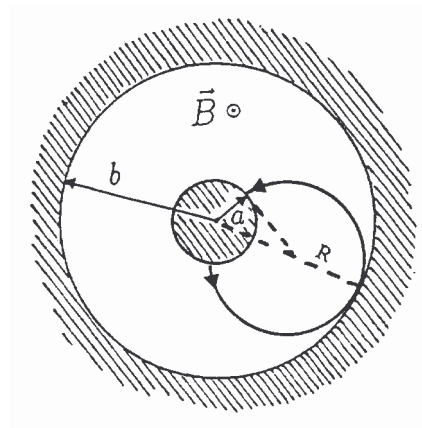
b) When  $V = 0$  the electron moves in a homogeneous static magnetic field. The magnetic Lorentz force acts orthogonal to the velocity and the electron will move in a circle. The initial velocity is tangential to the circle.

The radius  $R$  of the orbit (the "cyclotron radius") is determined by equating the centripetal force and the Lorentz force:

$$eBv_0 = \frac{mv_0^2}{R},$$

i.e.

$$B = \frac{mv_0}{eR} \quad (2)$$



From the figure we see that in the critical case the radius  $R$  of the circle satisfies

$$\sqrt{a^2 + R^2} = b - R$$

By squaring we obtain

$$a^2 + R^2 = b^2 - 2bR + R^2$$

i.e.

$$R = (b^2 - a^2) / 2b$$

Insertion of this value for the radius into the expression (2) gives the critical field

$$B_c = \frac{mv_0}{eR} = \frac{2bm v_0}{\underline{\underline{(b^2 - a^2)e}}}.$$

c) The change in angular momentum with time is produced by a torque. Here the azimuthal component  $F_\phi$  of the Lorentz force  $\vec{F} = (-e)\vec{B} \times \vec{v}$  provides a torque  $F_\phi r$ . It is only the radial component  $v_r = dr/dt$  of the velocity that provides an azimuthal Lorentz force.

Hence

$$\underline{\underline{\frac{dL}{dt} = eBr \frac{dr}{dt}}},$$

which can be rewritten as

$$\frac{d}{dt} \left( L - \frac{eBr^2}{2} \right) = 0.$$

Hence

$$C = \underline{\underline{L - \frac{1}{2} eBr^2}} \quad (3)$$

is constant during the motion. The dimensionless number  $k$  in the problem text is thus  $\underline{\underline{k = \frac{1}{2}}}$ .

d) We evaluate the constant  $C$ , equation (3), at the surface of the inner cylinder and at the maximal distance  $r_m$ :

$$0 - \frac{1}{2} eBa^2 = mvr_m - \frac{1}{2} eB r_m^2$$

which gives

$$v = \underline{\underline{\frac{eB(r_m^2 - a^2)}{2mr_m}}}. \quad (4)$$

**Alternative solution:** One may first determine the electric potential  $V(r)$  as function of the radial distance. In cylindrical geometry the field falls off inversely proportional to  $r$ , which requires a logarithmic potential,  $V(s) = c_1 \ln r + c_2$ . When the two constants are determined to yield  $V(a) = 0$  and  $V(b) = V$  we have

$$V(r) = V \frac{\ln(r/a)}{\ln(b/a)}.$$

The gain in potential energy,  $sV(r_m)$ , is converted into kinetic energy:

$$\frac{1}{2} mv^2 = eV \frac{\ln(r_m/a)}{\ln(b/a)}.$$

Thus

$$v = \sqrt{\frac{2eV}{m} \frac{\ln(r_m / a)}{\ln(b / a)}}. \quad (5)$$

(4) and (5) seem to be different answers. This is only apparent since  $r_m$  is not independent parameter, but determined by  $B$  and  $V$  so that the two answers are identical.

e) For the critical magnetic field the maximal distance  $r_m$  equals  $b$ , the radius of the outer cylinder, and the speed at the turning point is then

$$v = \frac{eB(b^2 - a^2)}{2mb}.$$

Since the Lorentz force does not work, the corresponding kinetic energy  $\frac{1}{2} m v^2$  equals  $eV$  (question a):

$$v = \sqrt{2eV/m}.$$

The last two equations are consistent when

$$\frac{eB(b^2 - a^2)}{2mb} = \sqrt{2eV/m}.$$

The critical magnetic field for current cut-off is therefore

$$B_c = \frac{2b}{b^2 - a^2} \sqrt{\frac{2mV}{e}}.$$

f) The Lorentz force has no component parallel to the magnetic field, and consequently the velocity component  $v_B$  is constant under the motion. The corresponding displacement parallel to the cylinder axis has no relevance for the question of reaching the anode.

Let  $v$  denote the final azimuthal speed of an electron that barely reaches the anode. Conservation of energy implies that

$$\frac{1}{2} m (v_B^2 + v_\phi^2 + v_r^2) + eV = \frac{1}{2} m (v_B^2 + v^2),$$

giving

$$v = \sqrt{v_r^2 + v_\phi^2 + 2eV/m}. \quad (6)$$

Evaluating the constant  $C$  in (3) at both cylinder surfaces for the critical situation we have

$$m v_\phi a - \frac{1}{2} e B_c a^2 = m v b - \frac{1}{2} e B_c b^2.$$

Insertion of the value (6) for the velocity  $v$  yields the critical field

$$B_c = \frac{2m(vb - v_\phi a)}{e(b^2 - a^2)} = \frac{2mb}{e(b^2 - a^2)} \left[ \sqrt{v_r^2 + v_\phi^2 + 2eV/m} - v_\phi a/b \right].$$



### **Solution Problem 3**

**a)** With the centre of the earth as origin, let the centre of mass  $C$  be located at  $\vec{l}$ . The distance  $l$  is determined by

$$M l = M_m (L - l),$$

which gives

$$l = \frac{M_m}{M + M_m} L = \underline{\underline{4.63 \cdot 10^6 \text{ m}}}, \quad (1)$$

less than  $R$ , and thus inside the earth.

The centrifugal force must balance the gravitational attraction between the moon and the earth:

$$M \omega^2 l = G \frac{M M_m}{L^2},$$

which gives

$$\omega = \sqrt{\frac{G M_m}{L^2 l}} = \sqrt{\frac{G(M + M_m)}{L^3}} = \underline{\underline{2.67 \cdot 10^{-6} \text{ s}^{-1}}}. \quad (2)$$

(This corresponds to a period  $2\pi/\omega = 27.2$  days.) We have used (1) to eliminate  $l$ .

**b)** The potential energy of the mass point  $m$  consists of three contributions:

(1) Potential energy because of rotation (in the rotating frame of reference, see the problem text),

$$-\frac{1}{2} m \omega^2 r_1^2,$$

where  $\vec{r}_1$  is the distance from  $C$ . This corresponds to the centrifugal force  $m \omega^2 r_1$ , directed outwards from  $C$ .

(2) Gravitational attraction to the earth,

$$-G \frac{mM}{r}.$$

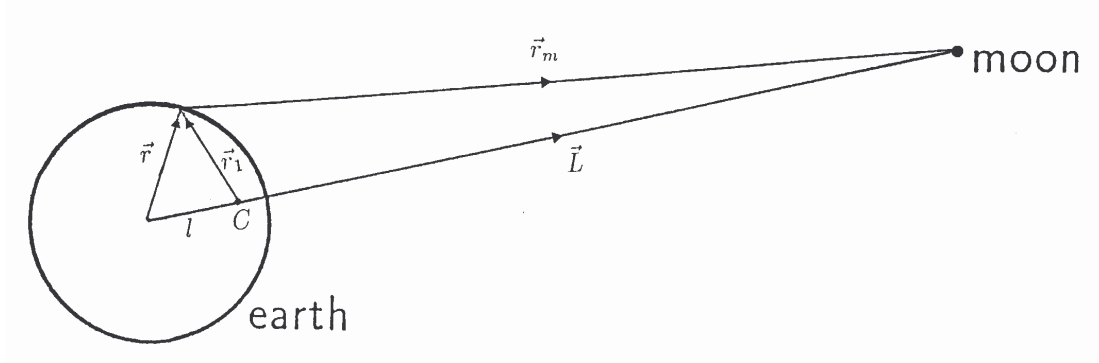
(3) Gravitational attraction to the moon,

$$-G \frac{mM_m}{r_m},$$

where  $\vec{r}_m$  is the distance from the moon.

Describing the position of  $m$  by polar coordinates  $r, \varphi$  in the plane orthogonal to the axis of rotation (see figure), we have

$$\vec{r}_1^2 = (\vec{r} - \vec{l})^2 = r^2 - 2rl\cos\varphi + l^2.$$



Adding the three potential energy contributions, we obtain

$$V(\vec{r}) = -\frac{1}{2}m\omega^2(r^2 - 2rl\cos\varphi + l^2) - G\frac{mM}{r} - G\frac{mM_m}{|\vec{L} - \vec{r}|}. \quad (3)$$

Here  $l$  is given by (1) and

$$|\vec{r}_m| = \sqrt{(\vec{L} - \vec{r})^2} = \sqrt{L^2 - 2\vec{L}\vec{r} + r^2} = L\sqrt{1 + (r/L)^2 - 2(r/L)\cos\varphi},$$

c) Since the ratio  $r/L = a$  is very small, we may use the expansion

$$\frac{1}{\sqrt{1 + a^2 - 2a\cos\varphi}} = 1 + a\cos\varphi + a^2\frac{1}{2}(3\cos^2\varphi - 1).$$

Insertion into the expression (3) for the potential energy gives

$$V(r, \varphi)/m = -\frac{1}{2}\omega^2 r^2 - \frac{GM}{r} - \frac{GM_m r^2}{2L^3}(3\cos^2\varphi - 1), \quad (4)$$

apart from a constant. We have used that

$$m\omega^2 rl\cos\varphi - GmM_m \frac{r}{L^2}\cos\varphi = 0,$$

when the value of  $\omega^2$ , equation (2), is inserted.

The form of the liquid surface is such that a mass point has the same energy  $V$  *everywhere on the surface*. (This is equivalent to requiring no net force tangential to the surface.) Putting

$$r = R + h,$$

where the tide  $h$  is much smaller than  $R$ , we have approximately

$$\frac{1}{r} = \frac{1}{R+h} = \frac{1}{R} \frac{1}{1+(h/R)} \cong \frac{1}{R} \left(1 - \frac{h}{R}\right) = \frac{1}{R} - \frac{h}{R^2},$$

as well as

$$r^2 = R^2 + 2Rh + h^2 \cong R^2 + 2Rh.$$

Inserting this, and the value (2) of  $\omega$  into (4), we have

$$V(r, \varphi)/m = -\frac{G(M+M_m)R}{L^3}h + \frac{GM}{R^2}h - \frac{GM_m r^2}{2L^3}(3\cos^2 \varphi - 1), \quad (5)$$

again apart from a constant.

The magnitude of the first term on the right-hand side of (5) is a factor

$$\frac{(M+M_m)}{M} \left(\frac{R}{L}\right)^3 \cong 10^{-5}$$

smaller than the second term, thus negligible. If the remaining two terms in equation (5) compensate each other, *i.e.*

$$h = \frac{M_m r^2 R^2}{2ML^3} (3\cos^2 \varphi - 1),$$

then the mass point  $m$  has the same energy everywhere on the surface. Here  $r^2$  can safely be approximated by  $R^2$ , giving the tidal bulge

$$h = \frac{M_m R^4}{2ML^3} (3\cos^2 \varphi - 1).$$

The largest value  $h_{\max} = M_m R^4 / ML^3$  occurs for  $\varphi = 0$  or  $\pi$ , in the direction of the moon or in the opposite direction, while the smallest value  $h_{\min} = -M_m R^4 / 2ML^3$  corresponds to  $\varphi = \pi/2$  or  $3\pi/2$ . The difference between high tide and low tide is therefore

$$h_{\max} - h_{\min} = \frac{3M_m R^4}{2ML^3} = \underline{\underline{0.54\text{m}}}.$$

(The values for high and low tide are determined up to an additive constant, but the difference is of course independent of this.)