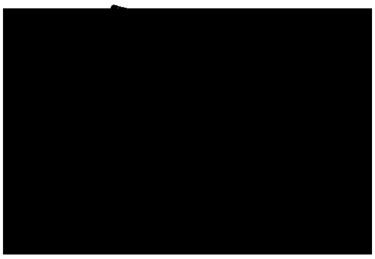
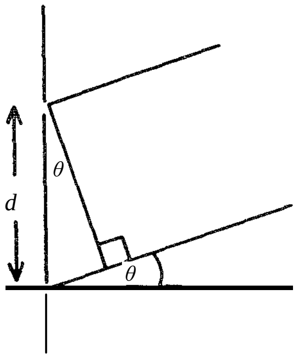
**Answers Question 1**

(i) Vector Diagram






If the phase of the light from the first slit is zero, the phase from second slit is

$$\phi = \frac{2\pi}{\lambda} d \sin\theta$$

Adding the two waves with phase difference $\phi$ where $\xi = 2\pi\left(ft - \frac{x}{\lambda}\right)$,

$$a\cos(\xi+\phi) + a\cos(\xi) = 2a\cos(\phi/2)\left(\xi+\phi/2\right)$$
$$a\cos(\xi+\phi) + a\cos(\xi) = 2a\cos\beta\left\{\cos\left(\xi+\beta\right)\right\}$$

This is a wave of amplitude $A = 2a\cos\beta$ and phase $\beta$. From vector diagram, in isosceles triangle OPQ,

$$\beta = \frac{1}{2}\phi = \frac{\pi}{\lambda} d \sin\theta \qquad (NB \quad \phi = 2\beta)$$

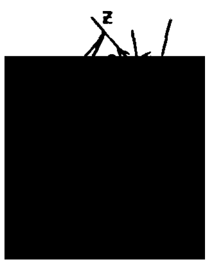
and

$$A = 2a\cos\beta.$$

Thus the sum of the two waves can be obtained by the addition of two vectors of amplitude a and angular directions 0 and $\phi$.

(ii) Each slit in diffraction grating produces a wave of amplitude a with phase $2\beta$ relative to previous slit wave. The vector diagram consists of a 'regular' polygon with sides of constant length $a$ and with constant angles between adjacent sides.
Let O be the centre of circumscribing circle passing through the vertices of the polygon. Then radial lines such as OS have length R and bisect the internal angles of the polygon. Figure 1.2.

Figure 1.2



$$O\hat{S}T = O\hat{T}S = \frac{1}{2}(180-\phi)$$

$$\text{and} \quad T\hat{O}S = \phi$$

In the triangle *TOS*, for example

$$a = 2R\sin(\phi/2) = 2R\sin\beta \text{ as } (\phi = 2\beta)$$

$$\therefore R = \frac{a}{2\sin\beta} \quad (1)$$

As the polygon has *N* faces then:

$$T\hat{O}Z = N(T\hat{O}Z) = N\phi = 2N\beta$$

Therefore in isosceles triangle TOZ, the amplitude of the resultant wave, TZ, is given by

$$2R\sin N\beta \, .$$

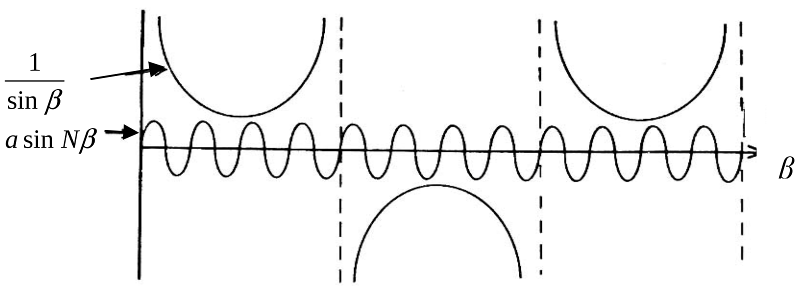
Hence form (1) this amplitude is

$$\frac{a\sin N\beta}{\sin\beta}$$

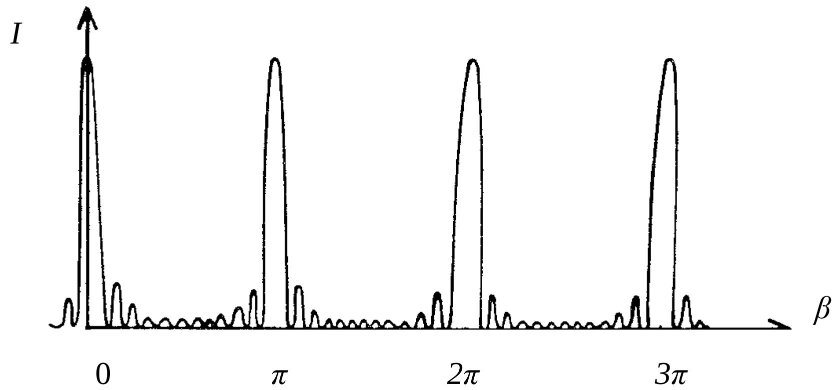
Resultant phase is

$$= Z\hat{T}S$$
$$= O\hat{T}S - O\hat{T}Z$$
$$\left(90-\frac{\phi}{2}\right)-\frac{1}{2}(180-N\phi)$$
$$-\frac{1}{2}(N-1)\phi$$
$$= (N-1)\beta$$

(iii)




Intensity $I = \dfrac{a^2\sin^2 N\beta}{\sin^2\beta}$




(iv) For the principle maxima $\beta = \pi p$ where $p = 0 \pm 1 \pm 2........$

$$I_{\max} = a^2\left(\frac{N\beta'}{\beta'}\right) = N^2 a^2 \qquad \beta' = 0 \text{ and } \beta = \pi p + \beta'$$

(v) Adjacent max. estimate $I_1$ :

$$\sin^2 N\beta = 1, \quad \beta = 2\pi p \mp \frac{3\pi}{2N} \text{ i.e } \beta = \pm\frac{3\pi}{2N}$$

$\left[\beta = \pi p \pm \frac{\pi}{2N}\right]$ does not give a maximum as can be observed from the graph.

$$I_1 = a^2 \frac{1}{\frac{3\pi^2}{2n}} = \frac{a^2 N^2}{23} \text{ for } N >> 1$$

Adjacent zero intensity occurs for $\beta = \pi\rho \pm \frac{\pi}{N}$ i.e. $\delta = \pm\frac{\pi}{N}$

For phase differences much greater than $\delta$, $\mathrm{I} = \mathrm{a}^2\left(\frac{\sin N\beta}{\sin\beta}\right) = a^2$ .

(vi)

$\beta = n\pi$ for a principle maximum

i.e. $\frac{\pi}{\lambda} d \sin\theta = n\pi \qquad n = 0, \pm 1, \pm 2..........$

Differentiating w.r.t, $\lambda$

$d \cos\theta \Delta\theta = n\Delta\lambda$

$$\Delta\theta = \frac{n\Delta\lambda}{d\cos\theta}$$

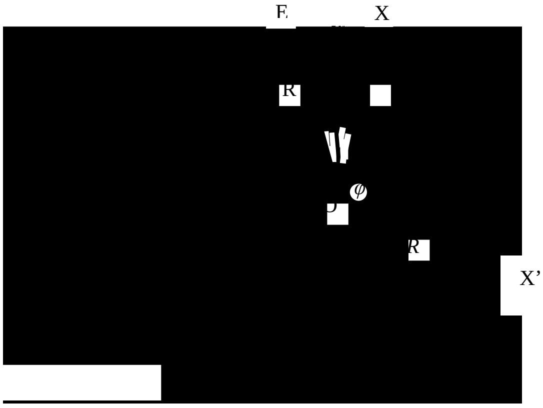
Substituting $\lambda = 589.0\text{nm}$, $\lambda + \Delta\lambda = 589.6\,\text{nm}$. n = 2 and $d = 1.2\times10^{-6}$ m.

$$\Delta\theta = \frac{n\Delta\lambda}{d\sqrt{1-\left(\frac{n\lambda}{d}\right)^2}} \quad \text{as } \sin\theta = \frac{n\lambda}{d} \text{ and } \cos\theta = \sqrt{1-\left(\frac{n\lambda}{d}\right)^2}$$

$\Rightarrow \Delta\theta = 5.2\times10^{-3}\,rads$ or $0.30^0$

**Answers Q2**

2.(i)




$$EX = 2R\sin\theta \quad \therefore t = \frac{2R\sin\theta}{v}$$

where $v = v_P$ for P waves and $v = v_S$ for S waves.

This is valid providing X is at an angular separation less than or equal to X', the tangential ray to the liquid core. X' has an angular separation given by, from the diagram,

$$2\phi = 2\cos^{-1}\left(\frac{R_C}{R}\right),$$

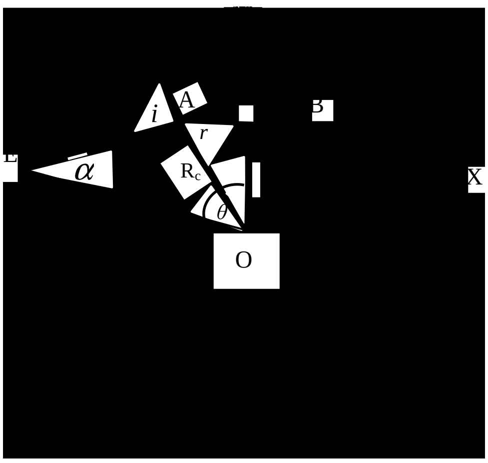
Thus

$$t = \frac{2R\sin\theta}{v} \quad , \qquad \text{for } \theta \le \cos^{-1}\left(\frac{R_C}{R}\right),$$

where $v = v_P$ for P waves and $v = v_S$ for shear waves.

(ii) $\frac{R_C}{R} = 0.5447$ and $\frac{v_{CP}}{v_P} = 0.831.3$

Figure 2.2




From Figure 2.2

$$\theta = A\hat{O}C + E\hat{O}A \Rightarrow \theta = (90 - r) + (1 - \alpha) \qquad (1)$$

(ii) Continued

Snell's Law gives:

$$\frac{\sin i}{\sin r} = \frac{v_P}{v_{CP}}. \qquad (2)$$

From the triangle EAO, sine rule gives

$$\frac{R_C}{\sin x} = \frac{R}{\sin i}. \qquad (3)$$

Substituting (2) and (3) into (1)

$$\theta = \left[ 90 - \sin^{-1}\left( \frac{v_{CP}}{v_P} \sin i \right) + i - \sin^{-1}\left( \frac{R_C}{R} \sin i \right) \right] \qquad (4)$$

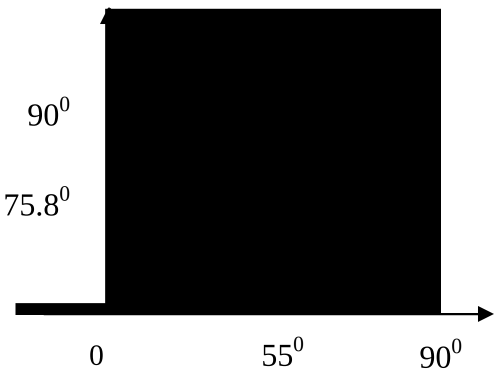
(iii)

<u>For Information Only</u>

For minimum $\theta, \frac{d\theta}{di} = 0$. $\Rightarrow 1 - \frac{\left(\frac{v_{CP}}{v_P}\right)\cos i}{\sqrt{1-\left(\frac{v_{CP}}{v_P}\sin i\right)^2}} - \frac{\left(\frac{R_C}{R}\right)\cos i}{\sqrt{1-\left(\frac{R_C}{R}\sin i\right)^2}} = 0$

Substituting $i = 55.0^{o}$ gives LHS=0, this verifying the minimum occurs at this value of $i$. Substituting $i = 55.0^{o}$ into (4) gives $\theta = 75.8^{0}$.

Plot of $\theta$ against $i$.




Substituting into 4:

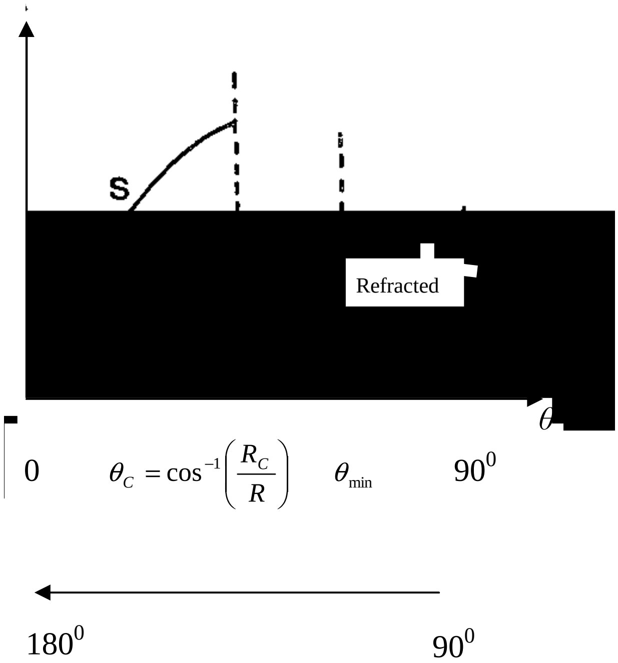
$i = 0$ gives $\theta = 90$

$i = 90°$ gives $\theta = 90.8°$

Substituting numerical values for $i = 0 \rightarrow 90°$ one finds a minimum value at $i = 55^{\circ}$; the minimum values of 0, $\theta_{MIN} = 75•8°$.

Physical Consequence

As $\theta$ has a minimum value of 75•8° observers at position for which 2 $\theta$ <151•6° will not observe the earthquake as seismic waves are not deviated by angles of less than 151•6°. However for 2 $\theta \leq$ 114° the direct, non-refracted, seismic waves will reach the observer.




(iv) Using the result

$$t = \frac{2r\sin\theta}{v}$$

the time delay $\Delta t$ is given by

$$\Delta t = 2R\sin\theta\left[\frac{1}{v_S} - \frac{1}{v_P}\right]$$

Substituting the given data

$$131 = 2(6370)\left[\frac{1}{6.31} - \frac{1}{10.85}\right]\sin\theta$$

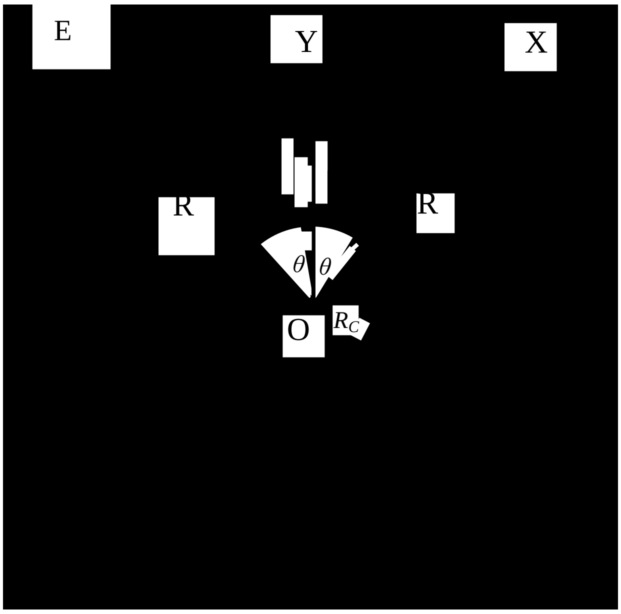
Therefore the angular separation of E and X is

$$2\theta = 17.84^o$$

This result is less than $2\cos^{-1}\left(\frac{R_C}{R}\right) = 2\cos^{-1}\left(\frac{3470}{6370}\right) = 114^o$

And consequently the seismic wave is not refracted through the core.

(v)




The observations are most likely due to reflections from the mantle-core interface. Using the symbols given in the diagram, the time delay is given by

$$\Delta t' = (\text{ED} + \text{DX})\left[\frac{1}{v_S} - \frac{1}{v_P}\right]$$

$$\Delta t' = 2(\text{ED})\left[\frac{1}{v_S} - \frac{1}{v_P}\right] \text{ as ED} = \text{EX by symmetry}$$

In the triangle EYD,

$$(\text{ED})^2 = (R\sin\theta)^2 + (R\cos\theta - R_C)^2$$

$$(\text{ED})^2 = R^2 + R_C^2 - 2RR_C\cos\theta \qquad \sin^2\theta + \cos^2\theta = 1$$

Therefore

$$\Delta t' = 2\sqrt{R^2 + R_C^2 - 2RR_C\cos\theta}\left[\frac{1}{v_S} - \frac{1}{v_P}\right]$$

Using (ii)

$$\Delta t' = \frac{\Delta t}{R\sin\theta}\sqrt{R^2 + {R_C}^2 - 2RR_C\cos\theta}$$

$$\Rightarrow 396.7s \text{ or } 6\text{m } 37\text{s}$$

Thus the subsequent time interval, produced by the reflection of seismic waves at the mantle core interface, is consistent with angular separation of $17.84^0$.

**Answer Q3**

Equations of motion:

$$m\frac{d^2u_1}{dt^2} = k(u_2 - u_1) + k(u_3 - u_1)$$
$$m\frac{d^2u_2}{dt^2} = k(u_3 - u_2) + k(u_1 - u_2)$$
$$m\frac{d^2u_3}{dt^2} = k(u_1 - u_3) + k(u_2 - u_3)$$

Substituting $u_n(t) = u_n(0)\cos\omega t$ and $\omega_o{}^2 = \frac{k}{m}$:

$$(2\omega_o{}^2 - \omega^2)u_1(0) - \omega_o{}^2 u_2(0) - \omega_o{}^2 u_3(0) = 0 \quad \text{(a)}$$
$$-\omega_o{}^2 u_1(0) + (2\omega_o{}^2 - \omega^2)u_2(0) - \omega_o{}^2 u_3(0) = 0 \quad \text{(b)}$$
$$-\omega_o{}^2 u_1(0) - \omega_o{}^2 u_2(0) + (2\omega_o{}^2 - \omega^2)u_3(0) = 0 \quad \text{(c)}$$

Solving for $u_1(0)$ and $u_2(0)$ in terms of $u_3(0)$ using (a) and (b) and substituting into (c) gives the equation equivalent to

$$(3\omega_o{}^2 - \omega^2)^2\,\omega^2 = 0$$
$$\omega^2 = 3\omega_o{}^2,\ \ 3\omega_o{}^2 \ \text{ and } \ 0$$
$$\omega = \sqrt{3}\omega_o,\ \sqrt{3}\omega_o \ \text{ and } 0$$

(ii) Equation of motion of the n'th particle:

$$m\frac{d^2u_n}{dt^2} = k(u_{1+n} - u_n) + k(u_{n-1} - u_n)$$
$$\frac{d^2u_n}{dt^2} = k(u_{1+n} - u_n) + \omega_o{}^2(u_{n-1} - u_n)$$

$n = 1,2\ldots\ldots N$

Substituting $u_n(t) = u_n(0)\sin\left(2ns\frac{\pi}{N}\right)\cos\omega_s t$

$$-\omega_s{}^2\left(\sin\left(2ns\frac{\pi}{N}\right)\right) = \omega_o{}^2\left[\sin\left(2(n+1)s\frac{\pi}{N}\right) - 2\sin\left(2ns\frac{\pi}{N}\right) + \sin\left(2(n-1)s\frac{\pi}{N}\right)\right]$$
$$-\omega_s{}^2\left(\sin\left(2ns\frac{\pi}{N}\right)\right) = 2\omega_o{}^2\left[\frac{1}{2}\sin\left(2(n+1)s\frac{\pi}{N}\right) + \sin\left(2ns\frac{\pi}{N}\right) - \frac{1}{2}\sin\left(2(n-1)s\frac{\pi}{N}\right)\right]$$
$$-\omega_s{}^2\left(\sin\left(2ns\frac{\pi}{N}\right)\right) = 2\omega_o{}^2\left[\sin\left(2ns\frac{\pi}{N}\right)\cos\left(2s\frac{\pi}{N}\right) - \sin\left(2ns\frac{\pi}{N}\right)\right]$$
$$\therefore \omega_s{}^2 = 2\omega_o{}^2\left[1 - \cos\left(2s\frac{\pi}{N}\right)\right]: \quad (s = 1,2,\ldots\ldots N)$$

As $2\sin^2\theta = 1 - \cos 2\theta$

This gives

$$\omega_s = 2\omega_o\sin\left(\frac{s\pi}{N}\right) \quad (s = 1,2,\ldots N)$$

$\omega_s$ can have values from 0 to $2\omega_o = 2\sqrt{\frac{k}{m}}$ when $N \to \infty$; corresponding to range $s = 1$ to $\frac{N}{2}$.

(iv) For s'th mode

$$\frac{u_n}{u_{n+1}} = \frac{\sin\left(2ns\frac{\pi}{N}\right)}{\sin\left(2(n+1)s\frac{\pi}{N}\right)}$$

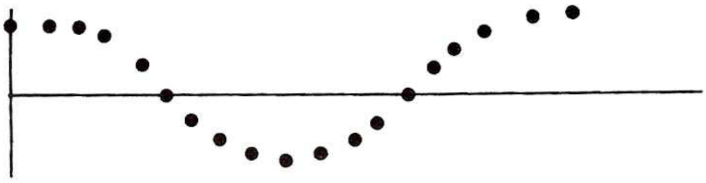
$$\frac{u_n}{u_{n+1}} = \frac{\sin\left(2ns\frac{\pi}{N}\right)}{\sin\left(2ns\frac{\pi}{N}\right)\cos\left(2s\frac{\pi}{N}\right)+\cos\left(2ns\frac{\pi}{N}\right)\sin\left(2s\frac{\pi}{N}\right)}$$

(a) For small $\omega$, $\left(\frac{s}{N}\right) \approx 0$, thus $\cos\left(2ns\frac{\pi}{N}\right) \cong 1$ and $= \sin\left(2ns\frac{\pi}{N}\right) \approx 0$, and so $\frac{u_n}{u_{n+1}} \cong 1$.

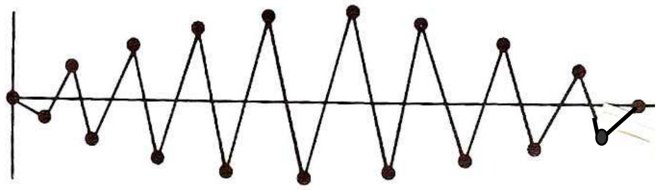
(b) The highest mode, $\omega_{\max} = 2\omega_o$, corresponds to $s = N/2$

$$\therefore \frac{u_n}{u_{n+1}} = -1 \quad \text{as} \quad \frac{\sin(2n\pi)}{\sin(2(n+1)\pi)} = -1$$

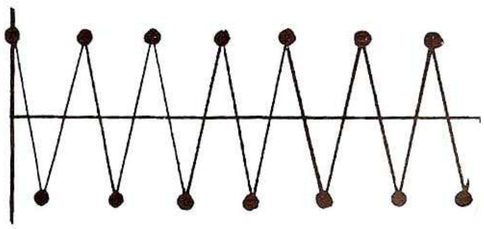
**Case (a)**



**Case (b)**
**N odd**

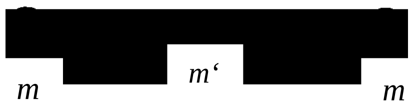


**N even**



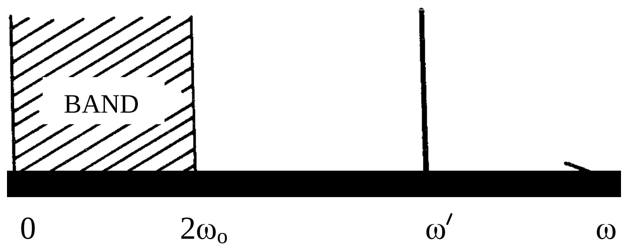
(vi) If $m' \ll m$, one can consider the frequency associated with $m'$ as due to vibration of $m'$ between two adjacent, much heavier, masses which can be considered stationary relative to $m'$.

The normal mode frequency of $m'$, in this approximation, is given by




$$m'\ddot{x} = -2kx$$
$$\omega'^2 = \frac{2k}{m}$$
$$\omega' = \sqrt{\frac{2k}{m'}}$$

For small $m'$, $\omega'$ will be much greater than $\omega_{max}$,




DIATOMIC SYSTEM

More light masses, $m'$, will increase the number of frequencies in region of $\omega'$ giving a band-gap-band spectrum.