3^{rd} International Physics Olympiad
1969, Brno, Czechoslovakia

Problem 1. Figure 1 shows a mechanical system consisting of three carts A, B and C of masses $m_1 = 0.3$ kg, $m_2 = 0.2$ kg and $m_3 = 1.5$ kg respectively. Carts B and A are connected by a light taut inelastic string which passes over a light smooth pulley attaches to the cart C as shown. For this problem, all resistive and frictional forces may be ignored as may the moments of inertia of the pulley and of the wheels of all three carts. Take the acceleration due to gravity g to be 9.81 m s^{-2}.

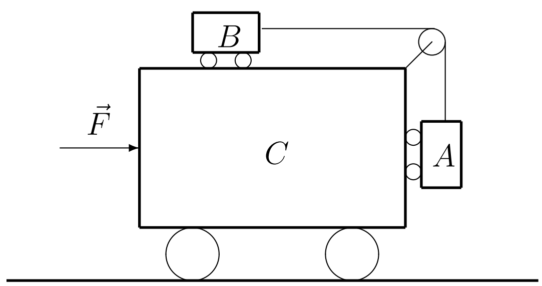

Figure 1:

1. A horizontal force $\vec{F}$ is now applied to cart C as shown. The size of $\vec{F}$ is such that carts A and B remain at rest relative to cart C.
   a) Find the tension in the string connecting carts A and B.
   b) Determine the magnitude of $\vec{F}$.
2. Later cart C is held stationary, while carts A and B are released from rest.
   a) Determine the accelerations of carts A and B.
   b) Calculate also the tension in the string.

Solution:
Case 1. The force $\vec{F}$ has so big magnitude that the carts A and B remain at the rest with respect to the cart C, *i.e.* they are moving with the same acceleration as the cart C is. Let $\vec{G_1}$, $\vec{T_1}$ and $\vec{T_2}$ denote forces acting on particular carts as shown in the Figure 2 and let us write the equations of motion for the carts A and B and also for whole mechanical system. Note that certain internal forces (viz. normal reactions) are not shown.

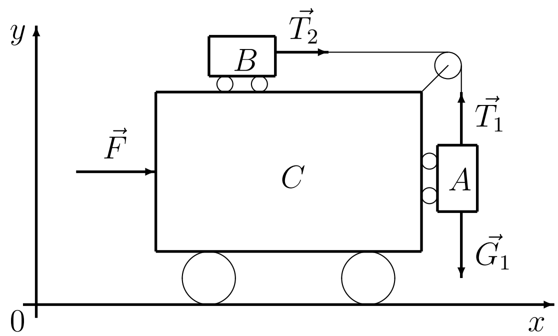

Figure 2:

The cart B is moving in the coordinate system Oxy with an acceleration a_x. The only force acting on the cart B is the force $\vec{T_2}$, thus

$$T_2 = m_2\, a_x\,. \tag{1}$$

Since $\vec{T_1}$ and $\vec{T_2}$ denote tensions in the same cord, their magnitudes satisfy

$$T_1 = T_2\,.$$

The forces $\vec{T_1}$ and $\vec{G_1}$ act on the cart A in the direction of the y-axis. Since, according to condition 1, the carts A and B are at rest with respect to the cart C, the acceleration in the direction of the y-axis equals to zero, $a_y = 0$, which yields

$$T_1 - m_1\, g = 0\,.$$

Consequently

$$T_2 = m_1\, g\,. \tag{2}$$

So the motion of the whole mechanical system is described by the equation

$$F = (m_1 + m_2 + m_3)\, a_x\,, \tag{3}$$

because forces between the carts A and C and also between the carts B and C are internal forces with respect to the system of all three bodies. Let us remark here that also the tension $\vec{T_2}$ is the internal force with respect to the system of all bodies, as can be easily seen from the analysis of forces acting on the pulley. From equations (1) and (2) we obtain

$$a_x = \frac{m_1}{m_2} g \, .$$

Substituting the last result to (3) we arrive at

$$F = (m_1 + m_2 + m_3) \, \frac{m_1}{m_2} g \, .$$

Numerical solution:

$$T_2 = T_1 = 0.3 \cdot 9.81 \text{ N} = 2.94 \text{ N} \, ,$$
$$F = 2 \cdot \frac{3}{2} \cdot 9.81 \text{ N} = 29.4 \text{ N} \, .$$

Case 2. If the cart C is immovable then the cart A moves with an acceleration a_y and the cart B with an acceleration a_x. Since the cord is inextensible (*i.e.* it cannot lengthen), the equality

$$a_x = -a_y = a$$

holds true. Then the equations of motion for the carts A, respectively B, can be written in following form

$$T_1 = G_1 - m_1 \, a \, , \tag{4}$$
$$T_2 = m_2 \, a \, . \tag{5}$$

The magnitudes of the tensions in the cord again satisfy

$$T_1 = T_2 \, . \tag{6}$$

The equalities (4), (5) and (6) immediately yield

$$(m_1 + m_2) \, a = m_1 \, g \, .$$

Using the last result we can calculate

$$a = a_x = -a_y = \frac{m_1}{m_1 + m_2}\, g\,,$$
$$T_2 = T_1 = \frac{m_2 m_1}{m_1 + m_2}\, g\,.$$

Numerical results:

$$a = a_x = \frac{3}{5} \cdot 9.81 \text{ m s}^{-2} = 5.89 \text{ m s}^{-2}\,,$$
$$T_1 = T_2 = 1.18 \text{ N}\,.$$

Problem 2. Water of mass m_2 is contained in a copper calorimeter of mass m_1. Their common temperature is t_2. A piece of ice of mass m_3 and temperature $t_3 < 0\,^\circ\text{C}$ is dropped into the calorimeter.

a) Determine the temperature and masses of water and ice in the equilibrium state for general values of m_1, m_2, m_3, t_2 and t_3. Write equilibrium equations for all possible processes which have to be considered.

b) Find the final temperature and final masses of water and ice for $m_1 = 1.00$ kg, $m_2 = 1.00$ kg, $m_3 = 2.00$ kg, $t_2 = 10\,^\circ\text{C}$, $t_3 = -20\,^\circ\text{C}$.

Neglect the energy losses, assume the normal barometric pressure. Specific heat of copper is $c_1 = 0.1$ kcal/kg·°C, specific heat of water $c_2 = 1$ kcal/kg·°C, specific heat of ice $c_3 = 0.492$ kcal/kg·°C, latent heat of fusion of ice $l = 78,7$ kcal/kg. Take 1 cal = 4.2 J.

Solution:
We use the following notation:

t	temperature of the final equilibrium state,
$t_0 = 0\,^\circ\text{C}$	the melting point of ice under normal pressure conditions,
M_2	final mass of water,
M_3	final mass of ice,
$m_2' \leq m_2$	mass of water, which freezes to ice,
$m_3' \leq m_3$	mass of ice, which melts to water.

a) Generally, four possible processes and corresponding equilibrium states can occur:

1. $t_0 < t < t_2$, $m_2' = 0$, $m_3' = m_3$, $M_2 = m_2 + m_3$, $M_3 = 0$.
   Unknown final temperature t can be determined from the equation

   $$(m_1c_1 + m_2c_2)(t_2 - t) = m_3c_3(t_0 - t_3) + m_3l + m_3c_2(t - t_0)\,. \qquad (7)$$

   However, only the solution satisfying the condition $t_0 < t < t_2$ does make physical sense.

2. $t_3 < t < t_0$, $m_2' = m_2$, $m_3' = 0$, $M_2 = 0$, $M_3 = m_2 + m_3$.
   Unknown final temperature t can be determined from the equation

   $$m_1c_1(t_2 - t) + m_2c_2(t_2 - t_0) + m_2l + m_2c_3(t_0 - t) = m_3c_3(t - t_3)\,. \quad (8)$$

   However, only the solution satisfying the condition $t_3 < t < t_0$ does make physical sense.

3. $t = t_0$, $m_2' = 0$, $0 \leq m_3' \leq m_3$, $M_2 = m_2 + m_3'$, $M_3 = m_3 - m_3'$.
   Unknown mass m_3' can be calculated from the equation

   $$(m_1c_1 + m_2c_2)(t_2 - t_0) = m_3c_3(t - t_3) + m_3'l\,. \qquad (9)$$

   However, only the solution satisfying the condition $0 \leq m_3' \leq m_3$ does make physical sense.

4. $t = t_0$, $0 \leq m_2' \leq m_2$, $m_3' = 0$, $M_2 = m_2 - m_2'$, $M_3 = m_3 + m_2'$.
   Unknown mass m_2' can be calculated from the equation

   $$(m_1c_1 + m_2c_2)(t_2 - t_0) + m_2'l = m_3c_3(t_0 - t_3)\,. \qquad (10)$$

   However, only the solution satisfying the condition $0 \leq m_2' \leq m_2$ does make physical sense.

b) Substituting the particular values of m_1, m_2, m_3, t_2 and t_3 to equations (7), (8) and (9) one obtains solutions not making the physical sense (not satisfying the above conditions for t, respectively m_3'). The real physical process under given conditions is given by the equation (10) which yields

$$m_2' = \frac{m_3c_3(t_0 - t_3) - (m_1c_1 + m_2c_2)(t_2 - t_0)}{l}\,.$$

Substituting given numerical values one gets $m_2' = 0.11$ kg. Hence, $t = 0\,^\circ$C, $M_2 = m_2 - m_2' = 0.89$ kg, $M_3 = m_3 + m_2' = 2.11$ kg.

Problem 3. A small charged ball of mass m and charge q is suspended from the highest point of a ring of radius R by means of an insulating cord of negligible mass. The ring is made of a rigid wire of negligible cross section and lies in a vertical plane. On the ring there is uniformly distributed charge Q of the same sign as q. Determine the length l of the cord so as the equilibrium position of the ball lies on the symmetry axis perpendicular to the plane of the ring.

Find first the general solution a then for particular values $Q = q = 9.0 \cdot 10^{-8}$ C, $R = 5$ cm, $m = 1.0$ g, $\varepsilon_0 = 8.9 \cdot 10^{-12}$ F/m.

Solution:
In equilibrium, the cord is stretched in the direction of resultant force of $\vec{G} = m\vec{g}$ and $\vec{F} = q\vec{E}$, where $\vec{E}$ stands for the electric field strength of the ring on the axis in distance x from the plane of the ring, see Figure 3. Using the triangle similarity, one can write

$$\frac{x}{R} = \frac{Eq}{mg} \,. \qquad (11)$$

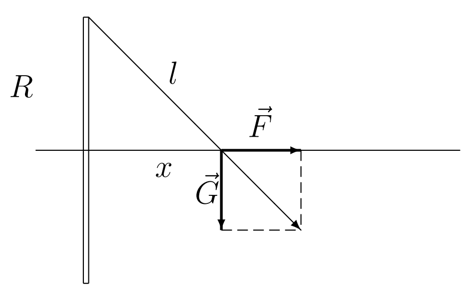

Figure 3:

For the calculation of the electric field strength let us divide the ring to n identical parts, so as every part carries the charge Q/n. The electric field strength magnitude of one part of the ring is given by

$$\Delta E = \frac{Q}{4\pi\varepsilon_0 l^2 n} \,.$$

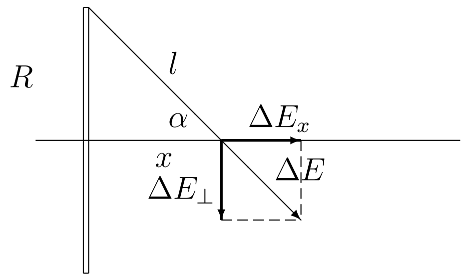

Figure 4:

This electric field strength can be decomposed into the component in the direction of the x-axis and the one perpendicular to the x-axis, see Figure 4. Magnitudes of both components obey

$$\Delta E_x = \Delta E \, \cos\alpha = \frac{\Delta E \, x}{l} \,,$$
$$\Delta E_\perp = \Delta E \, \sin\alpha \,.$$

It follows from the symmetry, that for every part of the ring there exists another one having the component $\Delta \vec{E_\perp}$ of the same magnitude, but however oppositely oriented. Hence, components perpendicular to the axis cancel each other and resultant electric field strength has the magnitude

$$E = E_x = n\Delta E_x = \frac{Q\,x}{4\pi\varepsilon_0\, l^3} \,. \qquad (12)$$

Substituting (12) into (11) we obtain for the cord length

$$l = \sqrt[3]{\frac{Q\,q\,R}{4\pi\varepsilon_0\, m\, g}} \,.$$

Numerically

$$l = \sqrt[3]{\frac{9.0\cdot 10^{-8}\cdot 9.0\cdot 10^{-8}\cdot 5.0\cdot 10^{-2}}{4\pi\cdot 8.9\cdot 10^{-12}\cdot 10^{-3}\cdot 9.8}}\ \text{m} = 7.2\cdot 10^{-2}\ \text{m}\,.$$

Problem 4. A glass plate is placed above a glass cube of 2 cm edges in such a way that there remains a thin air layer between them, see Figure 5.

Electromagnetic radiation of wavelength between 400 nm and 1150 nm (for which the plate is penetrable) incident perpendicular to the plate from above is reflected from both air surfaces and interferes. In this range only two wavelengths give maximum reinforcements, one of them is $\lambda = 400$ nm. Find the second wavelength. Determine how it is necessary to warm up the cube so as it would touch the plate. The coefficient of linear thermal expansion is $\alpha = 8.0 \cdot 10^{-6}\,{}^{\mathrm{o}}\mathrm{C}^{-1}$, the refractive index of the air $n = 1$. The distance of the bottom of the cube from the plate does not change during warming up.

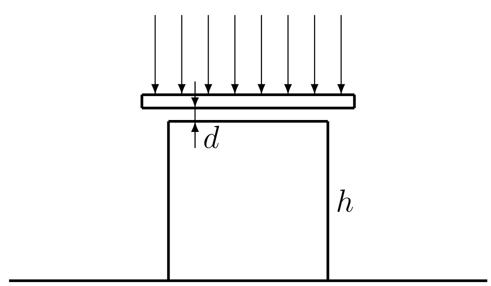

Figure 5:

Solution:
Condition for the maximum reinforcement can be written as

$$2dn - \frac{\lambda_k}{2} = k\lambda_k\,, \text{ for } k = 0, 1, 2, \dots\,,$$

i.e.

$$2dn = (2k+1)\frac{\lambda_k}{2}\,, \tag{13}$$

with d being thickness of the layer, n the refractive index and k maximum order. Let us denote $\lambda' = 1150$ nm. Since for $\lambda = 400$ nm the condition for maximum is satisfied by the assumption, let us denote $\lambda_p = 400$ nm, where p is an unknown integer identifying the maximum order, for which

$$\lambda_p(2p+1) = 4dn \tag{14}$$

holds true. The equation (13) yields that for fixed d the wavelength λ_k increases with decreasing maximum order k and vise versa. According to the assumption,

$$\lambda_{p-1} < \lambda' < \lambda_{p-2} \,,$$

i.e.

$$\frac{4dn}{2(p-1)+1} < \lambda' < \frac{4dn}{2(p-2)+1} \,.$$

Substituting to the last inequalities for $4dn$ using (14) one gets

$$\frac{\lambda_p(2p+1)}{2(p-1)+1} < \lambda' < \frac{\lambda_p(2p+1)}{2(p-2)+1} \,.$$

Let us first investigate the first inequality, straightforward calculations give us gradually

$$\lambda_p(2p+1) < \lambda'(2p-1) \,, \quad 2p(\lambda' - \lambda_p) > \lambda' + \lambda_p \,,$$

i.e.

$$p > \frac{1}{2}\,\frac{\lambda' + \lambda_p}{\lambda' - \lambda_p} = \frac{1}{2}\,\frac{1150+400}{1150-400} = 1.\,\ldots \tag{15}$$

Similarly, from the second inequality we have

$$\lambda_p(2p+1) > \lambda'(2p-3) \,, \quad 2p(\lambda' - \lambda_p) < 3\lambda' + \lambda_p \,,$$

i.e.

$$p < \frac{1}{2}\,\frac{3\lambda' + \lambda_p}{\lambda' - \lambda_p} = \frac{1}{2}\,\frac{3\cdot 1150+400}{1150-400} = 2.\,\ldots \tag{16}$$

The only integer p satisfying both (15) and (16) is $p = 2$.

Let us now find the thickness d of the air layer:

$$d = \frac{\lambda_p}{4}(2p+1) = \frac{400}{4}(2\cdot 2+1)\ \text{nm} = 500\ \text{nm}\,.$$

Substituting d to the equation (13) we can calculate λ_{p-1}, *i.e.* λ_1:

$$\lambda_1 = \frac{4dn}{2(p-1)+1} = \frac{4dn}{2p-1}.$$

Introducing the particular values we obtain

$$\lambda_1 = \frac{4\cdot 500\cdot 1}{2\cdot 2-1}\ \text{nm} = 666.7\ \text{nm}\,.$$

Finally, let us determine temperature growth Δt. Generally, $\Delta l = \alpha l \Delta t$ holds true. Denoting the cube edge by h we arrive at $d = \alpha h \Delta t$. Hence

$$\Delta t = \frac{d}{\alpha h} = \frac{5\cdot 10^{-7}}{8\cdot 10^{-6}\cdot 2\cdot 10^{-2}}\ {}^\circ\text{C} = 3.1\ {}^\circ\text{C}\,.$$