The Extremum Principle[1]

A. The Extremum Principle in Mechanics

Consider a horizontal frictionless x-y plane shown in Fig. 1. It is divided into two regions, I and II, by a line AB satisfying the equation $x = x_1$. The potential energy of a point particle of mass m in region I is $V = 0$ while it is $V = V_0$ in region II. The particle is sent from the origin O with speed v_1 along a line making an angle θ_1 with the x-axis. It reaches point P in region II traveling with speed v_2 along a line that makes an angle θ_2 with the $x-$axis. Ignore gravity and relativistic effects in this entire task T-2 (all parts).

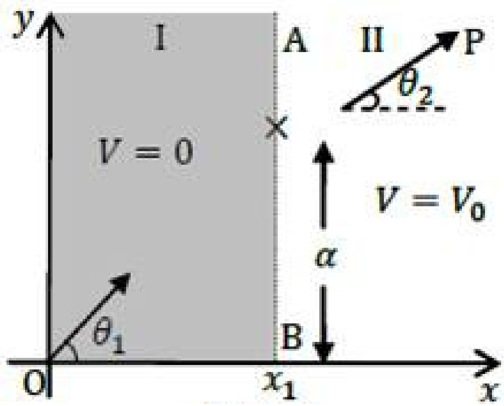


Figure 1

(A1) Obtain an expression for v_2 in terms of m, v_1 and V_0 . [0.2]

Solution:

From the principle of Conservation of Mechanical Energy

$$\frac{1}{2}mv_1^2 = \frac{1}{2}mv_2^2 + V_0$$

$$v_2 = (v_1^2 - \frac{2V_0}{m})^{1/2}$$

(A2) Express v_2 in terms of v_1, θ_1 and θ_2. [0.3]

Solution:

At the boundary there is an impulsive force ($\propto dV/dx$) in the $-x$ direction. Hence only the velocity component in the $x-$direction v_{1x} suffers change . The component in the $y-$direction remains unchanged. Therefore

$v_{1y} = v_{2y}$

$v_1 \sin\theta_1 = v_2 \sin\theta_2$

We define a quantity called action $A = m\int v(s)\, ds$, where ds is the infinitesimal length along the trajectory of a particle of mass m moving with speed $v(s)$. The integral is taken over the path. As an example. for a particle moving with constant speed v on a circular path of radius R, the action A for one revolution will be $2\pi mRv$. For a particle with constant energy E, it can be shown that of all the possible trajectories between two fixed points, the actual trajectory is the one on which A defined above is an extremum (minimum or maximum). Historically this is known as the Principle of Least Action (PLA).

[1] *Manoj Harbola (IIT-Kanpur) and Vijay A. Singh (ex-National Coordinator, Science Olympiads) were the principal authors of this problem. The contributions of the Academic Committee, Academic Development Group and the International Board are gratefully acknowledged.*

(A3) PLA implies that the trajectory of a particle moving between two fixed points in a region of constant potential will be a straight line. Let the two fixed points O and P in Fig. 1 have coordinates (0,0) and (x_0,y_0) respectively and the boundary point where the particle transits from region I to region II have coordinates (x_1,α). Note x_1 is fixed and the action depends on the coordinate α only. State the expression for the action $A(\alpha)$. Use PLA to obtain the the relationship between v_1/v_2 and these coordinates. [1.0]

Solution:

By definition $A(\alpha)$ from O to P is

$$A(\alpha) = mv_1\sqrt{x_1^2 + \alpha^2} + mv_2\sqrt{(x_0 - x_1)^2 + (y_0 - \alpha)^2}$$

Differentiating w.r.t. α and setting the derivative of $A(\alpha)$ to zero

$$\frac{v_1\alpha}{(x_1^2 + \alpha^2)^{1/2}} - \frac{v_2(y_0 - \alpha)}{[(x_0 - x_1)^2 + (y_0 - \alpha)^2]^{1/2}} = 0$$

$$\therefore \frac{v_1}{v_2} = \frac{(y_0 - \alpha)\,(x_1^2 + \alpha^2)^{1/2}}{\alpha\,[(x_0 - x_1)^2 + (y_0 - \alpha)^2]^{1/2}}$$

Note this is the same as A2, namely $v_1 \sin\theta_1 = v_2 \sin\theta_2$.

B. The Extremum Principle in Optics

A light ray travels from medium I to medium II with refractive indices n_1 and n_2 respectively. The two media are separated by a line parallel to the x-axis. The light ray makes an angle i_1 with the y-axis in medium I and i_2 in medium II (see Fig. 2). To obtain the trajectory of the ray, we make use of another extremum (minimum or maximum) principle known as Fermat's principle of least time.

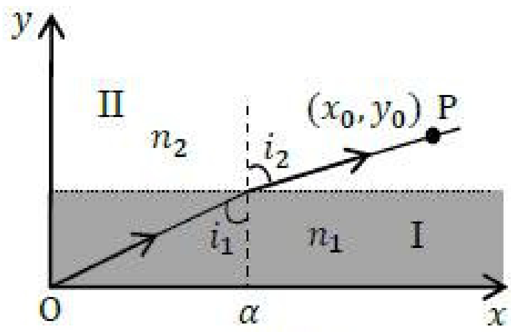


Figure 2

(B1) The principle states that between two fixed points, a light ray moves along a path such that the time taken between the two points is an extremum. Derive the relation between $\sin i_1$ and $\sin i_2$ on the basis of Fermat's principle. [0.5]

Solution:

The speed of light in medium I is c/n_1 and in medium II is c/n_2, where c is the speed of light in vacuum. Let the two media be separated by the fixed line $y = y_1$. Then time $T(\alpha)$ for light to travel from origin (0,0) and (x_0,y_0) is

$$T(\alpha) = n_1(\sqrt{y_1^2 + \alpha^2})/c + n_2(\sqrt{(x_0 - \alpha)^2 + (y_0 - y_1)^2})/c$$

Differentiating w.r.t. α and setting the derivative of $T(\alpha)$ to zero

$$\frac{n_1\alpha}{(y_1^2+\alpha^2)^{1/2}} - \frac{n_2(y_0-\alpha)}{[(x_0-\alpha)^2+(y_0-y_1)^2]^{1/2}} = 0$$

$$\therefore \quad n_1 \sin i_1 = n_2 \sin i_2$$

[Note: Derivation is similar to A3. This is Snell's law.]

Shown in Fig. 3 is a schematic sketch of the path of a laser beam incident horizontally on a solution of sugar in which the concentration of sugar decreases with height. As a consequence, the refractive index of the solution also decreases with height.

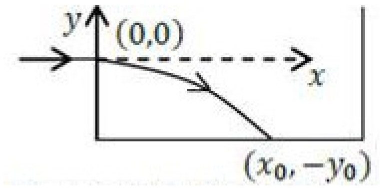


Figure 3

(B2) Assume that the refractive index n(y) depends only on y. Use the equation obtained in B1 to obtain the expresssion for the slope dy/dx of the beam's path in terms of n_0 at $y = 0$ and $n(y)$. [1.5]

Solution:

From Snell's law $n_0 \sin i_0 = n(y) \sin i$

Then, $$\frac{dy}{dx} = -\cot i$$

$$n_0 \sin i_0 = \frac{n(y)}{\sqrt{1+(\frac{dy}{dx})^2}}$$

$$\frac{dy}{dx} = -\sqrt{\left(\frac{n(y)}{n_0 \sin i_0}\right)^2 - 1}$$

(B3) The laser beam is directed horizontally from the origin (0,0) into the sugar solution at a height y_0 from the bottom of the tank as shown. Take $n(y) = n_0 - ky$ where n_0 and k are positive constants. Obtain an expression for x in terms of y and related quantities. You may use: $\int \sec\theta d\theta = \ln(\sec\theta + \tan\theta)$ + constant $\sec\theta = 1/\cos\theta$ or $\int \frac{dx}{\sqrt{x^2-1}} = \ln(x+\sqrt{x^2-1})$ + constant. [1.2]

Solution:

$$\int \frac{dy}{\sqrt{(\frac{n_0-ky}{n_0 \sin\ i_0})^2 - 1}} = -\int dx$$

Note $i_0 = 90^o$ so $\sin i_0 = 1$.

Method I We employ the substitution

$$\xi = \frac{n_0 - ky}{n_0}$$

$$\int \frac{d\xi(-\frac{n_0}{k})}{\sqrt{\xi^2 - 1}} = -\int dx$$

Let $\xi = \sec\theta$. Then

$$\frac{n_0}{k}\ln(\sec\theta + \tan\theta) = x + c$$

Or METHOD II

We employ the substitition

$$\xi = \frac{n_0 - ky}{n_0}$$

$$\int \frac{d\xi(-\frac{n_0}{k})}{\sqrt{\xi^2 - 1}} = -\int dx$$

$$-\frac{n_0}{k}\ln\left(\frac{n_0 - ky}{n_0} + \sqrt{(\frac{n_0 - ky}{n_0})^2 - 1}\right) = -x + c$$

Now continuing

Considering the substitutions and boundary condition, $x = 0$ for $y = 0$ we obtain that the constant $c = 0$.
Hence we obtain the following trajectory:

$$x = \frac{n_0}{k}\ln\left(\frac{n_0 - ky}{n_0} + \sqrt{(\frac{n_0 - ky}{n_0})^2 - 1}\right)$$

(B4) Obtain the value of x_0, the point where the beam meets the bottom of the tank. Take y_0 $= 10.0$ cm, $n_0 = 1.50$, $k = 0.050$ cm^{-1} (1 cm $= 10^{-2}$ m). [0.8]

Solution:

Given $y_0 = 10.0$ cm. $n_0 = 1.50$ $k = 0.050$ cm^{-1}
From (B3)

$$x_0 = \frac{n_0}{k}\ln\left[\left(\frac{n_0 - ky}{n_0}\right) + \left(\left(\frac{n_0 - ky}{n_0}\right)^2 - 1\right)^{1/2}\right]$$

Here $y = -y_0$

$$
\begin{aligned}
x_0 &= \frac{n_0}{k} \ln \left[\frac{(n_0 + ky_0)}{n_0} + \left(\frac{(n_0 + ky_0)^2}{n_0^2} - 1 \right)^{1/2} \right] \\
&= 30 \ln \left[\frac{2}{1.5} + \left(\left(\frac{2}{1.5} \right)^2 - 1 \right)^{1/2} \right] \\
&= 30 \ln \left[\frac{4}{3} + \left(\frac{7}{9} \right)^{1/2} \right] \\
&= 30 \ln \left[\frac{4}{3} + 0.88 \right] \\
&= 24.0 \text{ cm}
\end{aligned}
$$

C. The Extremum Principle and the Wave Nature of Matter

We now explore between the PLA and the wave nature of a moving particle. For this we assume that a particle moving from O to P can take all possible trajectories and we will seek a trajectory that depends on the constructive interference of de Broglie waves.

(C1) As the particle moves along its trajectory by an infinitesimal distance Δs, relate the change $\Delta\phi$ in the phase of its de Broglie wave to the change ΔA in the action and the Planck constant. [0.6]

Solution:

From the de Broglie hypothesis

$$\lambda \to \lambda_{dB} = h/mv$$

where λ is the de Broglie wavelength and the other symbols have their usual meaning

$$
\begin{aligned}
\Delta\phi &= \frac{2\pi}{\lambda}\Delta s \\
&= \frac{2\pi}{h} mv\Delta s \\
&= \frac{2\pi\Delta A}{h}
\end{aligned}
$$

(C2) Recall the problem from part A where the particle traverses from O to P (see Fig. 4). Let an opaque partition be placed at the boundary AB between the two regions. There is a small opening CD of width d in AB such that $d \ll (x_0 - x_1)$ and $d \ll x_1$. Consider two extreme paths OCP and ODP such that OCP lies on the classical trajectory discussed in part A. Obtain the phase difference $\Delta\phi_{CD}$ between the two paths to first order. [1.2]

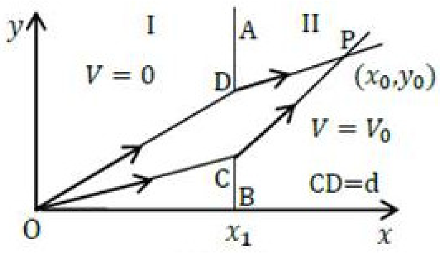
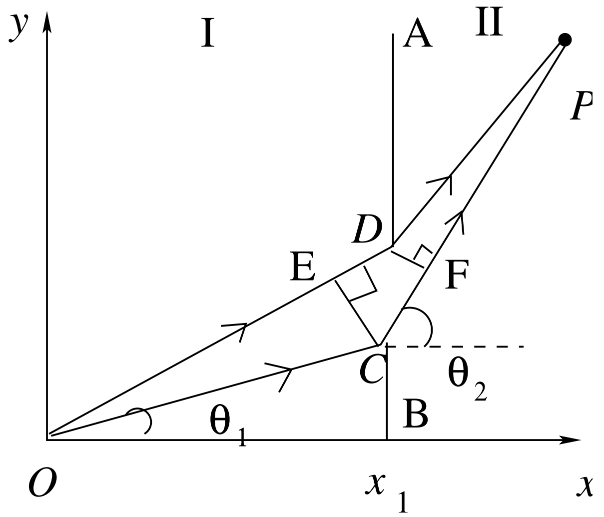


Figure 4

Solution:



Consider the extreme trajectories OCP and ODP of (C1)
The geometrical path difference is ED in region I and CF in region II.
This implies (note: $d \ll (x_0 - x_1)$ and $d \ll x_1$)

$$\Delta\phi_{CD} = \frac{2\pi d \sin\theta_1}{\lambda_1} - \frac{2\pi d \sin\theta_2}{\lambda_2}$$

$$\Delta\phi_{CD} = \frac{2\pi m v_1 d \sin\theta_1}{h} - \frac{2\pi m v_2 d \sin\theta_2}{h}$$

$$= 2\pi \frac{md}{h}(v_1 \sin\theta_1 - v_2 \sin\theta_2)$$

$$= 0 \qquad \text{(from A2 or B1)}$$

Thus near the clasical path there is invariably constructive interference.

D. Matter Wave Interference

Consider an electron gun at O which directs a collimated beam of electrons to a narrow slit at F in the opaque partition A_1B_1 at $x = x_1$ such that OFP is a straight line. P is a point on the screen at $x = x_0$ (see Fig. 5). The speed in I is $v_1 = 2.0000 \times 10^7\ \mathrm{m\,s^{-1}}$ and $\theta = 10.0000°$. The potential in region II is such that the speed $v_2 = 1.9900 \times 10^7\ \mathrm{m\,s^{-1}}$. The distance $x_0 - x_1$ is 250.00 mm (1 mm = 10^{-3} m). Ignore electron-electron interaction.

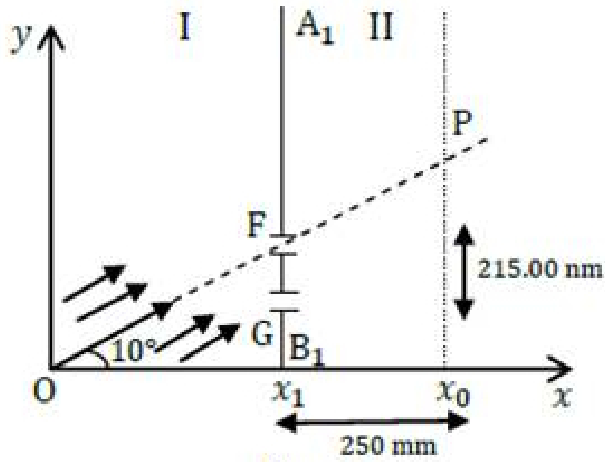


Figure 5

(D1) If the electrons at O have been accelerated from rest, calculate the accelerating potential U_1. [0.3]

Solution:

$$qU_1 = \frac{1}{2}\,mv^2$$

$$= \frac{9.11 \times 10^{-31} \times 4 \times 10^{14}}{2} J$$

$$= 2 \times 9.11 \times 10^{-17} J$$

$$= \frac{2 \times 9.11 \times 10^{-17}}{1.6 \times 10^{-19}}\ eV$$

$$= 1.139 \times 10^3\ eV \quad (\simeq 1100\ eV)$$

$$U_1 = 1.139 \times 10^3\ V$$

(D2) Another identical slit G is made in the partition A_1B_1 at a distance of 215.00 nm (1 nm = 10^{-9} m) below slit F (Fig. 5). If the phase difference between de Broglie waves ariving at P from F and G is $2\,\pi\,\beta$, calculate β. [0.8]

Solution: Phase difference at P is

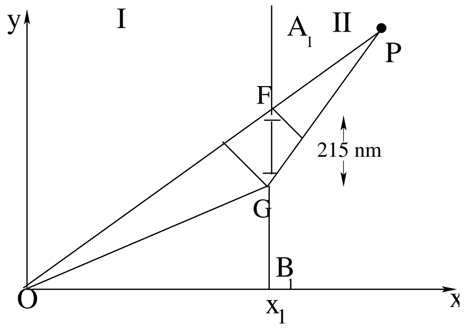
$$\Delta\phi_P = \frac{2\pi d \sin\theta}{\lambda_1} - \frac{2\pi d \sin\theta}{\lambda_2}$$

$$= 2\pi(v_1 - v_2)\frac{md}{h} \sin 10° = 2\pi\beta$$

$$\beta = 5.13$$

(D3) What is is the smallest distance Δy from P at which null (zero) electron detection maybe expected on the screen? [Note: you may find the approximation $\sin(\theta+\Delta\theta) \approx \sin\theta + \Delta\theta\cos\theta$ useful] [1.2]

Solution:



From previous part for null (zero) electron detection $\Delta\phi = 5.5 \times 2\pi$

$$\therefore mv_1\frac{d\sin\theta}{h} - \frac{mv_2 d\sin(\theta+\Delta\theta)}{h} = 5.5$$

$$\begin{aligned} sin(\theta+\Delta\theta) &= \frac{\frac{mv_1 d\sin\theta}{h} - 5.5}{\frac{mv_2 d}{h}} \\ &= \frac{v_1}{v_2}\sin\theta - \frac{h}{m}\frac{5.5}{v_2 d} \\ &= \frac{2}{1.99}\sin 10^\circ - \frac{5.5}{1374.78 \times 1.99 \times 10^7 \times \times 2.15 \times 10^{-7}} \\ &= 0.174521 - 0.000935 \end{aligned}$$

This yields $\Delta\theta = -0.0036^\circ$

The closest distance to P is

$$\begin{aligned} \Delta y &= (x_0 - x_1)(\tan(\theta+\Delta\theta) - \tan\theta) \\ &= 250(\tan 9.9964 - \tan 10) \\ &= -0.0162mm \\ &= -16.2\mu\,\mathrm{m} \end{aligned}$$

The negative sign means that the closest minimum is below P.

Approximate Calculation for θ and Δy

Using the approximation $\sin(\theta+\Delta\theta) \approx \sin\theta + \Delta\theta\cos\theta$

The phase difference of $5.5 \times 2\pi$ gives

$$mv_1\frac{d\sin 10^\circ}{h} - mv_2\frac{d(\sin 10^\circ + \Delta\theta\cos 10^\circ)}{h} = 5.5$$

From solution of the previous part

$$mv_1\frac{d\sin 10^\circ}{h} - mv_2\frac{dsin10^\circ}{h} = 5.13$$

Therefore

$$mv_2 \frac{d\Delta\theta \cos 10^\circ}{h} = 0.3700$$

This yields $\Delta\theta \approx 0.0036^\circ$
$\Delta y = -0.0162\,\text{mm} = -16.2\mu\text{m}$ as before

(D4) The electron beam has a square cross section of 500 nm × 500 nm and the setup is 2 m long. What should be the minimum beam flux density I_{min} (number of electrons per unit normal area per unit time) if, on an average, there is at least one electron in the setup at a given time? [0.4]

Solution: The product of the speed of the electrons and number of electron per unit volume on an average yields the intensity.
Thus $N = 1 =$ Intensity × Area × Length/ Electron Speed
$= I_{min} \times 0.25 \times 10^{-12} \times 2/2 \times 10^7$
This gives $I_{min} = 4\times\ 10^{19}\ \text{m}^{-2}\ \text{s}^{-1}$

R. Bach, D. Pope, Sy-H Liou and H. Batelaan, New J. of Physics Vol. 15, 033018 (2013).